

# **3강 유한요소법**

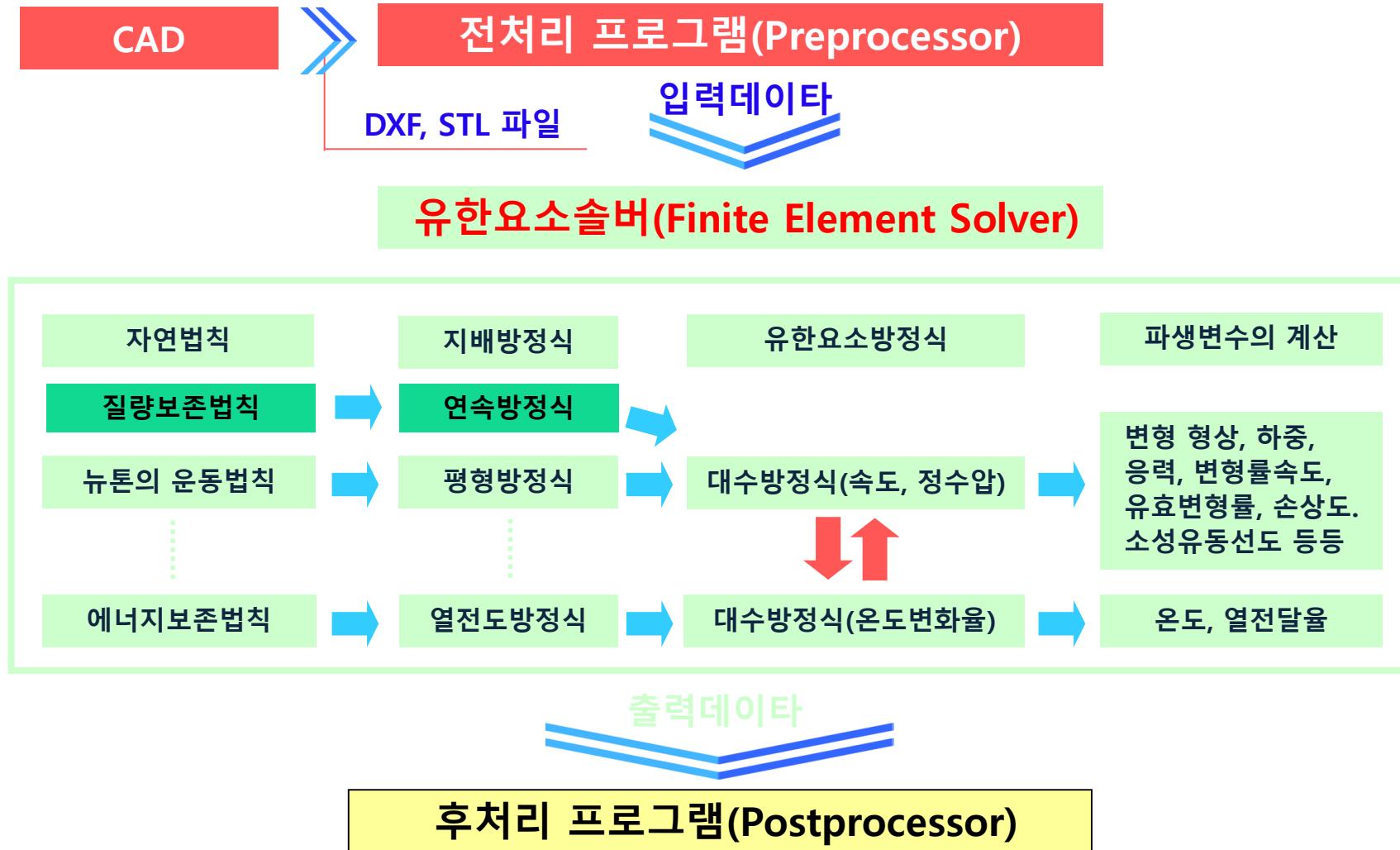
### 3강 목차

- 3.1 미분방정식의 근사해법 - Ritz법
- 3.2 미분방정식의 근사해법 - 가중오차법
- 3.3 유한요소법 개념
- 3.4 편미분방정식의 유한요소법

1.

## 미분방정식의 근사해법-Ritz법

# ▣ 시뮬레이션의 흐름도



▣ 자연현상(역학, 전자기학 등) = 경계치문제(미분방정식과 경계조건)

수학

▣ 경계치문제의 근사해법 = Ritz법과 가중오차법(Galerkin법)



유한요소법

실용적 근사해법

단조 시뮬레이션

▣ 단조 시뮬레이션 = 단조공정의 해  
석(성형해석+온도해석+금속학적  
해석)

## 경계치문제

미분방정식

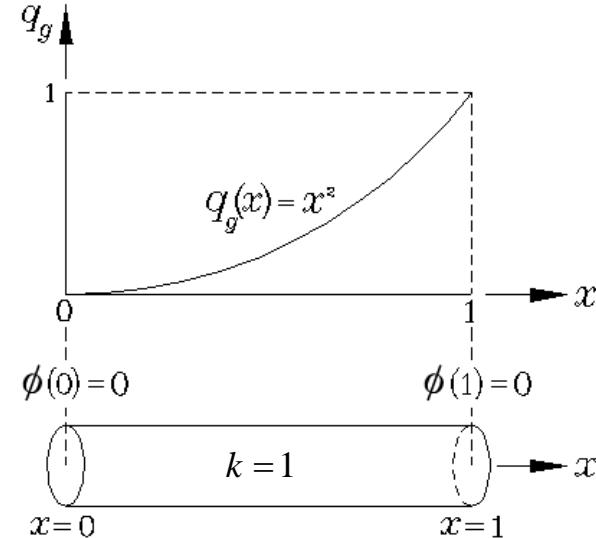
Prob. 1     $-\frac{d^2\phi}{dx^2} = x^2, \quad 0 < x < 1$

경계조건  
 $\phi(0) = 0, \quad \phi(1) = 0$

$$\phi''(x) = -x^2 \Rightarrow \phi'(x) = -\frac{1}{12}x^4 + C_1x + C_2$$

○ 정해 :  $\phi^*(x) = -\frac{1}{12}(x^4 - x)$

## 역학적 의미 부여



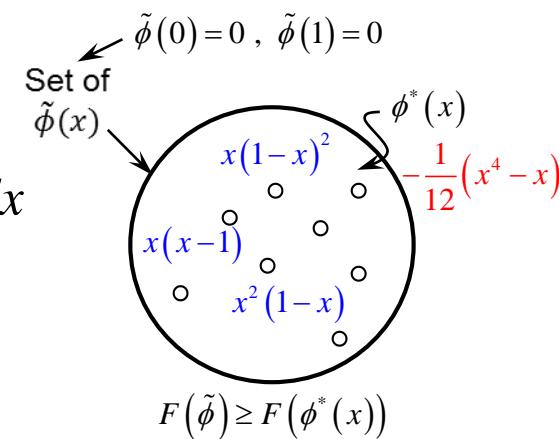
범함수(Functional)

## 변분원리

$$\phi = x \rightarrow F(\phi) = \frac{1}{2} \int_0^1 [1 - 2x^3] dx = \frac{1}{4}$$

Prob. 2    Extremize  $F(\phi) = \frac{1}{2} \int_0^1 \left[ \left( \frac{d\phi}{dx} \right)^2 - 2x^2\phi(x) \right] dx$

subject to  $\phi(0) = 0, \quad \phi(1) = 0$

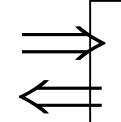


## 함수의 극화와 이와 관련된 대수방정식

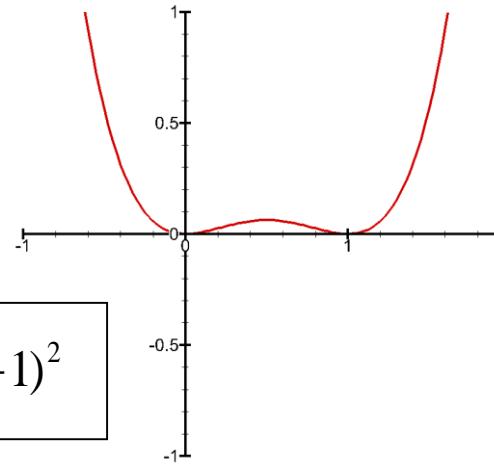
$$2x^3 - 3x^2 + x = 0$$



$$y' = x(x-1)(2x-1) = 0$$



$$\text{Extremize } y = x^2(x-1)^2$$

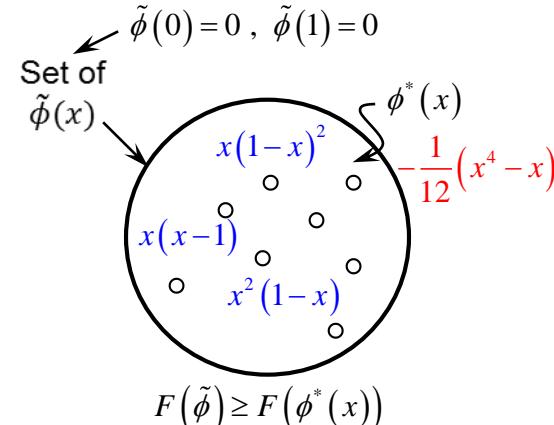


## 범함수의 극화와 이와 관련된 미분방정식 + 경계조건

$$-\frac{d^2\phi}{dx^2} = x^2, \quad 0 < x < 1, \quad \phi(0) = 0, \quad \phi(1) = 0$$



$$\begin{aligned} \text{Extremize } F(\phi) &= \frac{1}{2} \int_0^1 \left[ \left( \frac{d\phi}{dx} \right)^2 - 2x^2\phi(x) \right] dx \\ \phi(0) &= 0, \quad \phi(1) = 0 \end{aligned}$$



# Ritz 법 - 근사화

$$\tilde{\phi}(x) = C_1 x(1-x) + C_2 x^2(1-x)$$

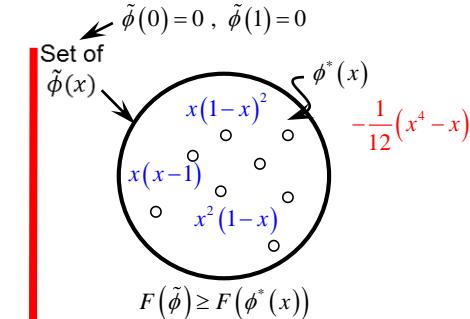
시도함수(Trial function)

$$\tilde{\phi}(x) = \sum_i C_i \zeta_i(x)$$

기초함수(Basic function)

$$\text{Extremize } F(\phi) = \frac{1}{2} \int_0^1 \left[ \left( \frac{d\phi}{dx} \right)^2 - 2x^2 \phi(x) \right] dx$$

subject to  $\phi(0) = 0, \phi(1) = 0$



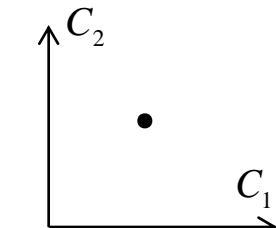
## 변환

함수장(Function space)  $\Rightarrow$  유한차원 벡터장(Finite dimensional vector space)

$$\begin{aligned} \circ F(\tilde{\phi}) &= \frac{1}{2} \int_0^1 \left[ \left\{ C_1(1-2x) + C_2(2x-3x^2) \right\}^2 - 2C_1 x^3(1-x) - 2C_2 x^4(1-x) \right] dx \\ &= \frac{1}{2} \int_0^1 (1-2x)^2 dx \cdot C_1^2 + \frac{1}{2} \int_0^1 (2x-3x^2)^2 dx \cdot C_2^2 \\ &\quad + \int_0^1 (1-2x)(2x-3x^2) dx \cdot C_1 C_2 - \int_0^1 x^3(1-x) dx C_1 - \int_0^1 x^4(1-x) dx C_2 \end{aligned}$$

$$\circ F(C_1, C_2) \equiv F(\tilde{\phi}) = \frac{1}{6} C_1^2 + \frac{1}{15} C_2^2 + \frac{1}{6} C_1 C_2 - \frac{1}{20} C_1 - \frac{1}{30} C_2$$

함수장  
=무한차원 벡터장



## ▣ 근사화 요약

$$\text{Extremize } F(\phi) = \frac{1}{2} \int_0^1 \left[ \left( \frac{d\phi}{dx} \right)^2 - 2x^2\phi(x) \right] dx$$

subject to  $\phi(0) = 0, \phi(1) = 0$

$\downarrow$   
 $\tilde{\phi}(0) = 0, \tilde{\phi}(1) = 0$

$$\tilde{\phi}(x) = C_1 x (1-x) + C_2 x^2 (1-x)$$

$$\text{Extremize } \tilde{F}(C_1, C_2) \equiv F(\tilde{\phi}) = \frac{1}{6} C_1^2 + \frac{1}{15} C_2^2 + \frac{1}{6} C_1 C_2 - \frac{1}{20} C_1 - \frac{1}{30} C_2$$

$$\frac{1}{3} C_1 + \frac{1}{6} C_2 = \frac{1}{20}$$

$$\frac{1}{6} C_1 + \frac{2}{15} C_2 = \frac{1}{30}$$

## ▣ 근사해의 계산

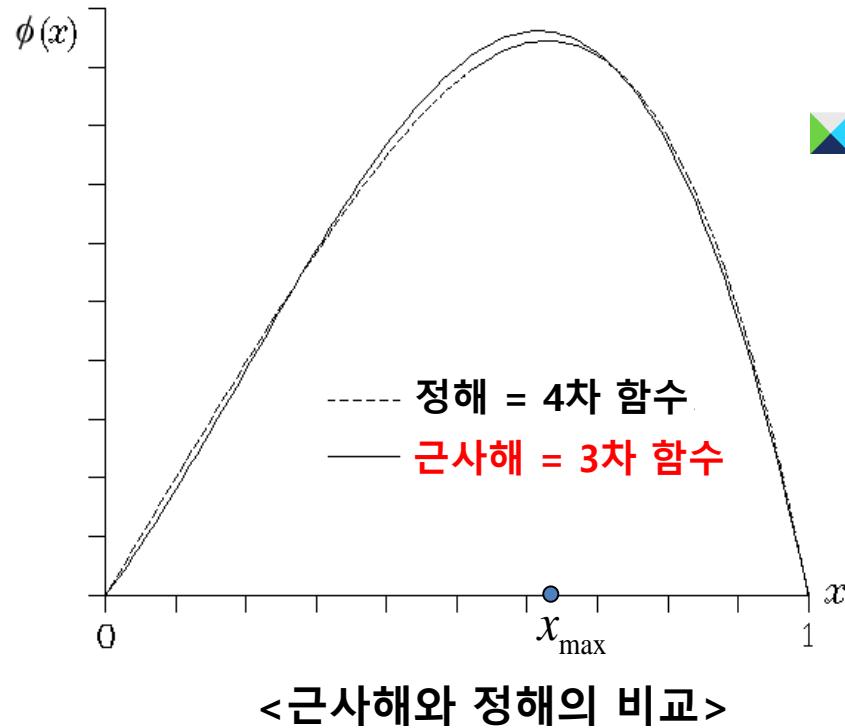
○ 함수  $\tilde{F}(C_1, C_2)$  가 극값을 가질 조건:  $\nabla \tilde{F} = \mathbf{0}$      $\rightarrow$      $\frac{\partial \tilde{F}}{\partial C_1} = 0, \quad \frac{\partial \tilde{F}}{\partial C_2} = 0$

○ 선형방정식 :  $\frac{1}{30} \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow C_1 = \frac{1}{15}, C_2 = \frac{1}{6}$

○ 근사해 :  $\tilde{\phi}(x) = \frac{1}{30} (2x + 3x^2 - 5x^3)$      $\longleftrightarrow$     정해  $\phi^*(x) = -\frac{1}{12} (x^4 - x)$

## ▣ 근사해와 정해의 비교

- $\phi^*(x_{\max}) = 0.029165, \tilde{\phi}(x_{\max}) = 0.029799 \Rightarrow \text{오차 } 2.2\%$
- $\phi^{**}(0) = \frac{1}{12}, \tilde{\phi}'(0) = \frac{1}{15}, \phi^{**}(1) = -\frac{1}{4}, \tilde{\phi}'(1) = -\frac{7}{30} \Rightarrow \text{오차 } 20\%$



### ▣ 해의 수렴 특성

- $\tilde{\phi}(x) = \sum_i C_i \zeta_i(x)$
- 정확도  $(\sum_{i=1}^n C_i \zeta_i(x)) < \text{정확도} (\sum_{i=1}^{n+1} C_i \zeta_i(x))$
- $\tilde{\phi}(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n C_i \zeta_i(x) \Rightarrow \phi^*(x)$

Accuracy ( $\tilde{\phi}_1(x)$ ) < Accuracy ( $\tilde{\phi}_2(x)$ ) < Accuracy ( $\tilde{\phi}_3(x)$ )

$$\tilde{\phi}_1(x) = C_1 x(1-x) + C_2 x^2(1-x)$$

$$\tilde{\phi}_2(x) = C_1 x(1-x) + C_2 x^2(1-x) + C_3 x^2(1-x)^2$$

$$\tilde{\phi}_3(x) = C_1 x(1-x) + C_2 x^2(1-x) + C_3 x^2(1-x)^2 + C_4 \sin\{x(1-x)\}$$

- 시도함수  $\tilde{\phi}(x) = \sum_i C_i \zeta_i(x)$ 가 정답을 표현할 수 있다면, 근사해=정해

2.

## 미분방정식의 근사해법 -가중오차법

(Weighted residual approach to differential equation)

# ◀ 약형의 유도

**Prob. 1**  $\frac{d^2\phi}{dx^2} + x^2 = 0, \quad 0 < x < 1$

$\phi(0) = 0, \quad \phi(1) = 0$

$\omega(x)$  is arbitrary.

$\omega(x)$  : 가중함수(Weighting function)

$$\int_a^b u'v = uv \Big|_a^b - \int_a^b uv'$$

$$\phi'(x)\omega(x) \Big|_0^1 - \int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0$$

$\phi(0) = \phi(1) = 0$

Assume  $\omega(0) = \omega(1) = 0$

## 약형(Weak form)

$$\left[ \int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0 \right]$$

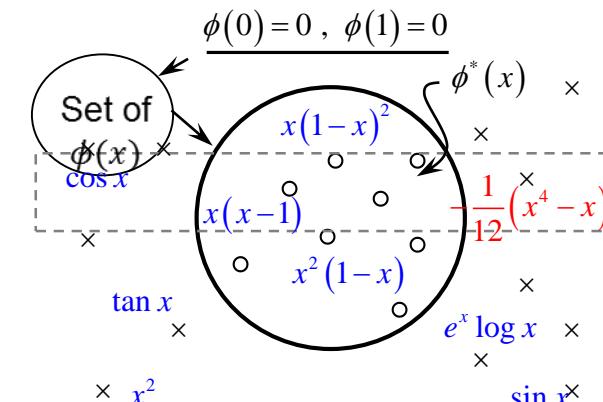
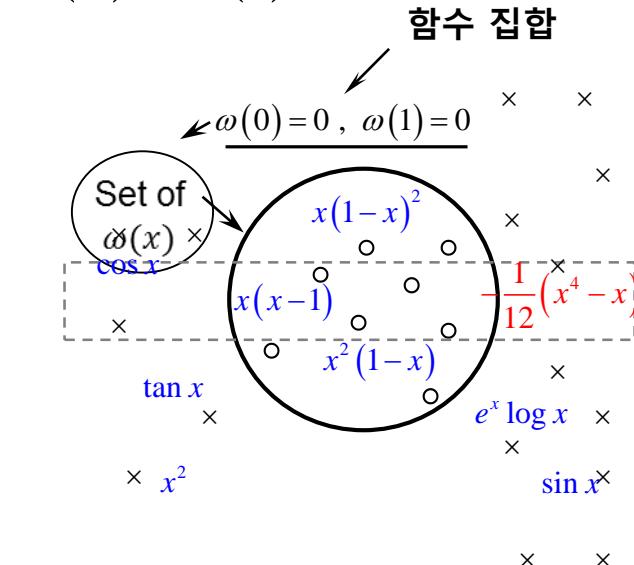
$\phi(0) = 0, \quad \phi(1) = 0$

where  $\omega(x)$  is arbitrary except that

$\omega(0) = 0$  and  $\omega(1) = 0$

**Prob. 2**  $\int_0^1 \omega(x) \left[ \frac{d^2\phi}{dx^2} + x^2 \right] dx = 0$

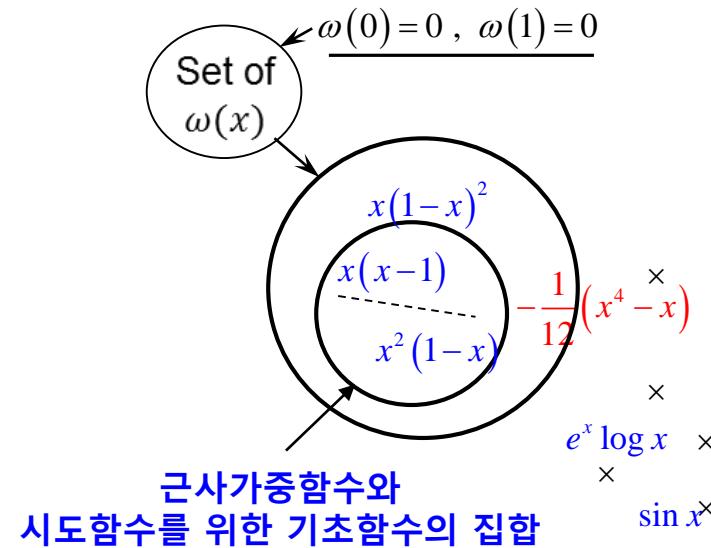
$\phi(0) = \phi(1) = 0$



## Galerkin 법

$$\left[ \int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0 \right. \\ \left. \phi(0) = 0, \phi(1) = 0 \right. \\ \left. \text{where } \omega(x) \text{ is arbitrary except that} \right. \\ \left. \omega(0) = 0 \text{ and } \omega(1) = 0 \right]$$

미지함수  $\phi(x)$  와 가중함수  $\omega(x)$  를  
동일한 기초함수로 근사화시키는 방법



## 미지함수 $\phi(x)$ 와 가중함수 $\omega(x)$ 의 근사화

$$\tilde{\phi}(x) = C_1 x(1-x) + C_2 x^2(1-x) ; \quad C_1, C_2 = \text{미지의 상수}$$

$$\tilde{\omega}(x) = W_1 x(1-x) + W_2 x^2(1-x) ; \quad W_1, W_2 = \text{임의의 상수}$$

근사가중함수  $\tilde{\omega}(0) = \tilde{\omega}(1) = 0$

$\left. \begin{array}{l} \zeta_1(x) = x(1-x) \\ \zeta_2(x) = x^2(1-x) \end{array} \right\}$

기초함수  
(Basic function)

$$\int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0$$

$$\int_0^1 [ \{C_1(1-2x) + C_2(2x-3x^2)\} \{W_1(1-2x) + W_2(2x-3x^2)\} dx - W_1 x^3(1-x) - W_2 x^4(1-x) ] dx = 0$$

## ▣ 항등식의 유도

$$\int_0^1 \left[ \{C_1(1-2x) + C_2(2x-3x^2)\} \{W_1(1-2x) + W_2(2x-3x^2)\} dx - W_1 x^3(1-x) - W_2 x^4(1-x) \right] dx = 0$$

$$W_1 \left[ \int_0^1 (1-2x)(1-2x)dx C_1 + \int_0^1 (1-2x)(2x-3x^2)dx C_2 - \int_0^1 x^3(1-x)dx \right] \stackrel{\downarrow}{\equiv \Phi_1} \quad \int_0^1 x^5 dx = \frac{1}{6}, \int_0^1 x^4 dx = \frac{1}{5}, \text{etc.}$$

$$+ W_2 \left[ \int_0^1 (2x-3x^2)(1-2x)dx C_1 + \int_0^1 (2x-3x^2)(2x-3x^2)dx C_2 - \int_0^1 x^4(1-x)dx \right] = 0 \stackrel{\uparrow}{\equiv \Phi_2}$$

$$W_1 \Phi_1 + W_2 \Phi_2 = 0$$

↑ ***W<sub>1</sub>*와 *W<sub>2</sub>*는 임의의 상수**

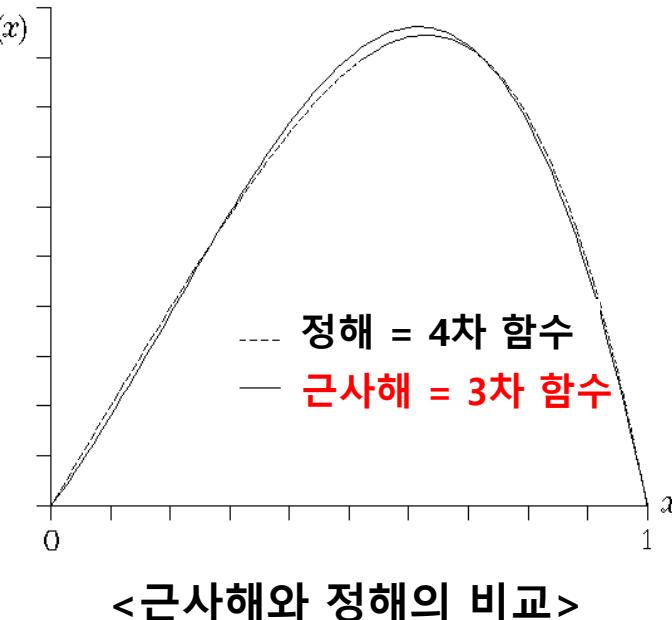
## ▣ 선형방정식

$$\frac{1}{30} \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow C_1 = \frac{1}{15}, C_2 = \frac{1}{6}$$

$$\tilde{\phi}(x) = C_1 x(1-x) + C_2 x^2(1-x)$$

**근사해 :**  $\tilde{\phi}(x) = \frac{1}{30} (2x + 3x^2 - 5x^3)$

↑ **Ritz법과 동일**



3.

## 유한요소법 개념

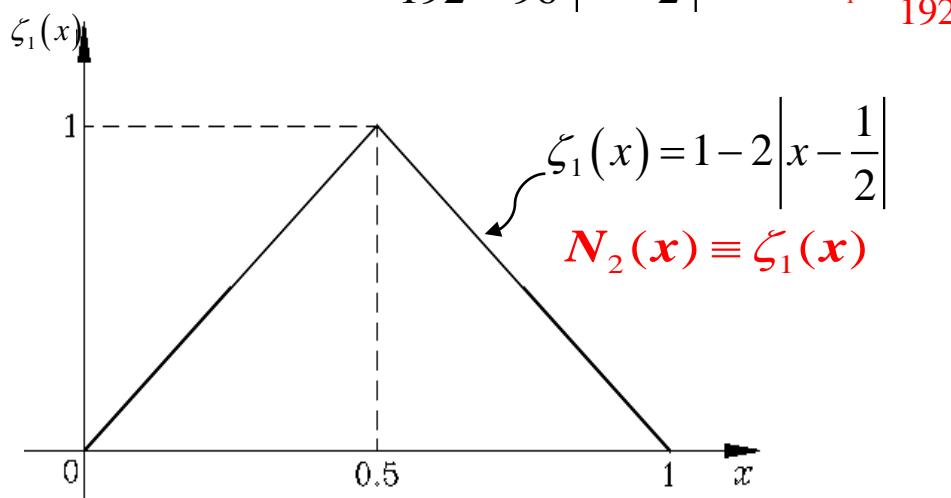
(Concept of variational approach to finite element  
method(FEM))

# 변분법적 유한요소법(FEM)의 기본 아이디어: Ritz 법에서 시작

$$\tilde{\phi}(x) = C_1 x (1-x) + C_2 x^2 (1-x)$$

↓ 단순화

- 시도함수 :  $\tilde{\phi}(x) = C_1 \left(1 - 2 \left|x - \frac{1}{2}\right|\right) = \begin{cases} 2C_1 x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2C_1 x(1-x) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$
- 근사해 :  $\tilde{\phi}(x) = \frac{7}{192} - \frac{7}{96} \left|x - \frac{1}{2}\right|$



<기초함수 = 보간함수(Interpolation function)>

요소(Element)

1 ① 2 ② 3 절점(Node)

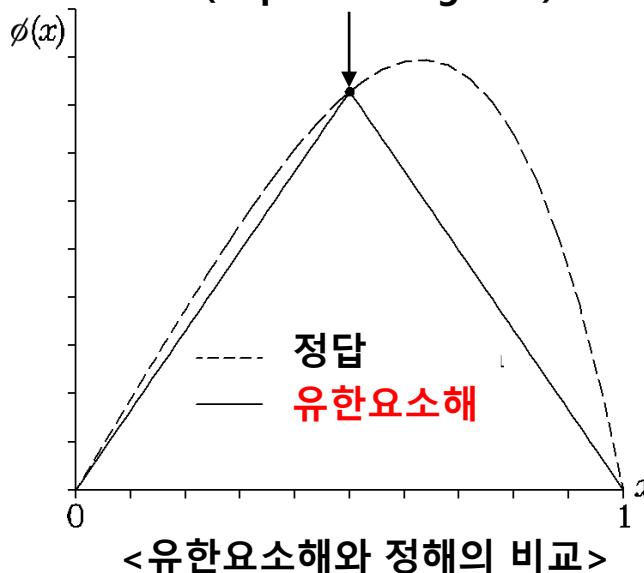
$$\text{Extremize } F(\phi) = \frac{1}{2} \int_0^1 \left[ \left( \frac{d\phi}{dx} \right)^2 - 2x^2 \phi(x) \right] dx$$

subject to  $\phi(0) = 0, \phi(1) = 0$

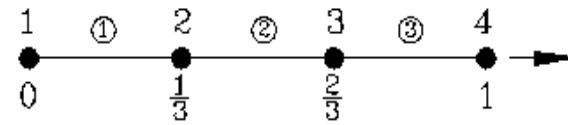
$C_1 = ?$

$$\tilde{\phi}(0.5) = C_1$$

초수렴  
(Superconvergence)



## 4절점 유한요소모델

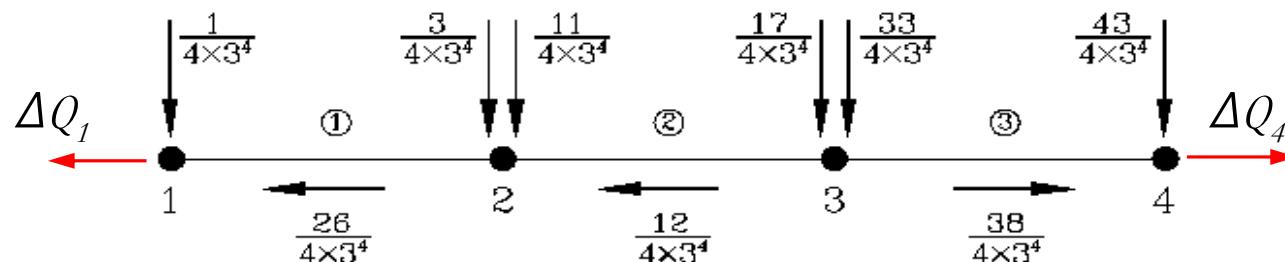


○ 요소 ①

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \frac{1}{4 \times 3^4} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

○ 요소 ③

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{bmatrix} = \frac{1}{4 \times 3^4} \begin{bmatrix} 33 \\ 43 \end{bmatrix}$$

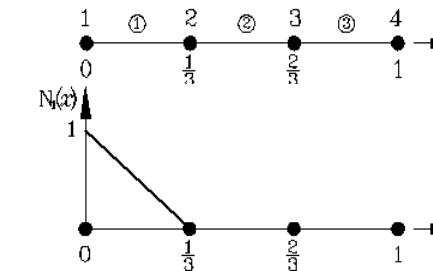


## 유한요소방정식

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \frac{1}{4 \times 3^5} \begin{bmatrix} 14 \\ 50 \end{bmatrix} \quad \left( \phi_1 = 0, \phi_2 = \frac{13}{486}, \phi_3 = \frac{19}{486}, \phi_4 = 0 \right)$$

## 보간함수

$$\phi(x) = \phi_1 N_1(x) + \phi_2 N_2(x) + \phi_3 N_3(x) + \phi_4 N_4(x)$$



경계조건

$$(\phi(0) = 0, \phi(1) = 0)$$

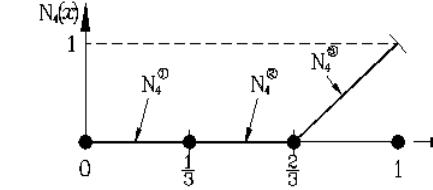
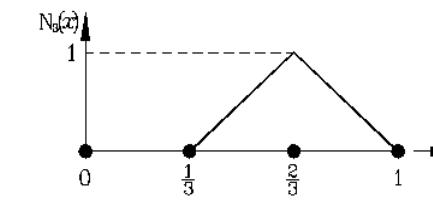
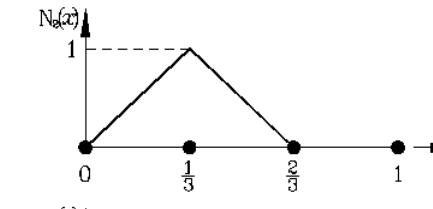
절점치

$$\rightarrow (\phi_1 = 0, \phi_4 = 0)$$

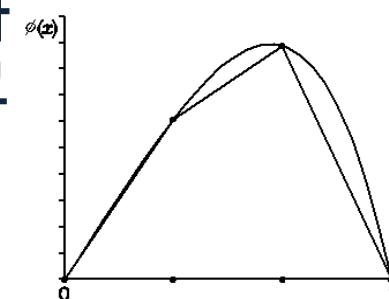
○ 요소 ②

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{bmatrix} = \frac{1}{4 \times 3^4} \begin{bmatrix} 11 \\ 17 \end{bmatrix}$$

요소방정식



## 근사해와 정해의 비교



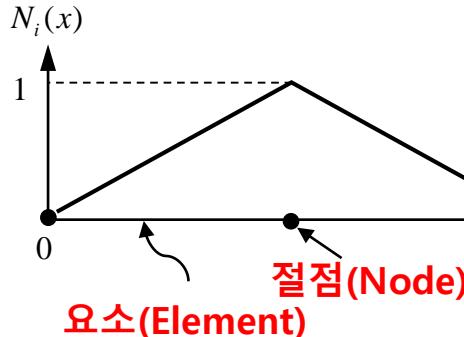
# FEM(유한요소법)

= Ritz 법/ Galerkin 법(미분방정식의 근사해법) + FE 이산화와 FE 보간(근사화)

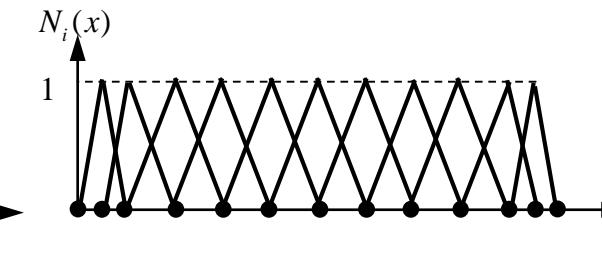
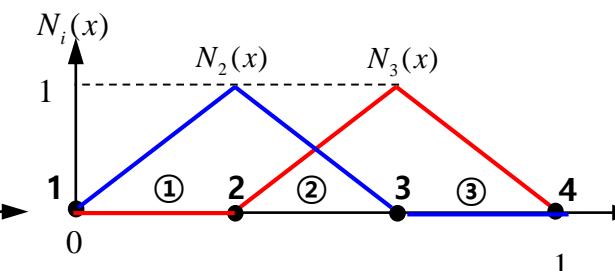
해석영역

미지함수

FE 보간함수 :  $N_i(x)$



기초함수를 만드는 방법 = 유한요소기교(Finite element technique)

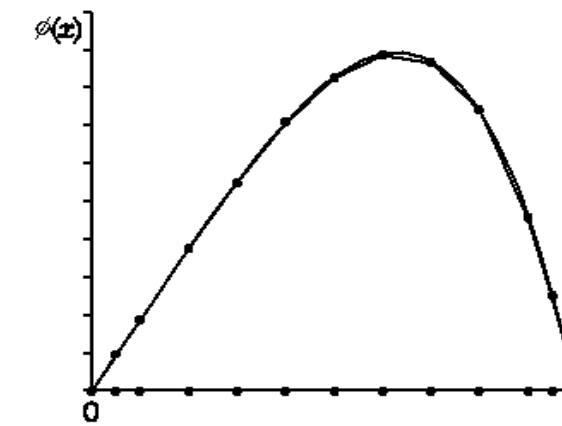
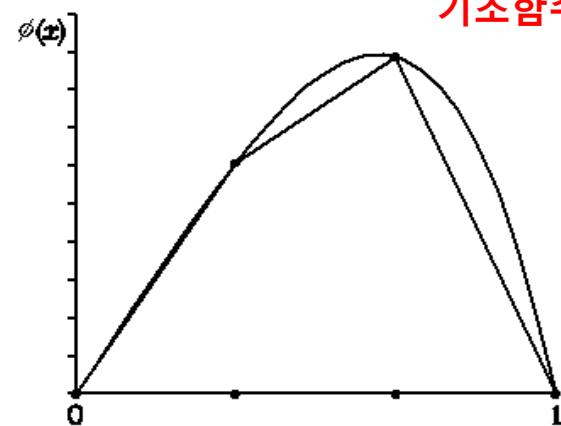
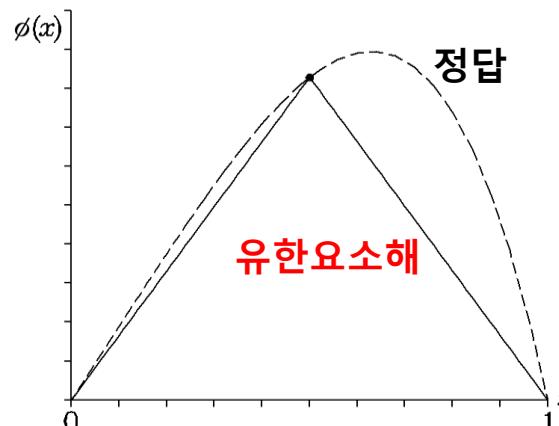


$$\tilde{\phi}(x) = \phi_2 N_2(x) + \phi_3 N_3(x) \quad \text{유한요소법}$$

$$\tilde{\phi}(x) = C_1 \zeta_1(x) + C_2 \zeta_2(x) \quad \text{Ritz 법과 Galerkin 법}$$

보간함수

유한요소해



# 4. 편미분방정식의 유한요소 법



## Poisson 방정식

$$\frac{\partial}{\partial \mathbf{x}} \left( k \frac{\partial \phi}{\partial \mathbf{x}} \right) + \frac{\partial}{\partial \mathbf{y}} \left( k \frac{\partial \phi}{\partial \mathbf{y}} \right) + f(x, y) = 0$$

$$T = \bar{T} \text{ on } S_T$$

1차원

$$-\frac{d^2 \phi}{dx^2} = x^2, \quad 0 < x < 1$$

$$\phi(0) = 0, \quad \phi(1) = 0$$

2차원 = 3차원

$$\text{Extremize } F(\phi) = \frac{1}{2} \iint \left[ k \left( \frac{\partial \phi}{\partial \mathbf{x}} \right)^2 + k \left( \frac{\partial \phi}{\partial \mathbf{y}} \right)^2 - 2f(x, y)\phi(x, y) \right] dxdy$$

$$\text{subject to } T = \bar{T} \text{ on } S_T$$

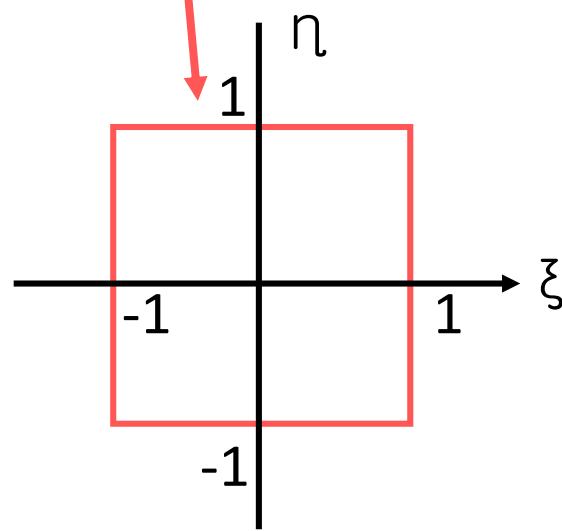
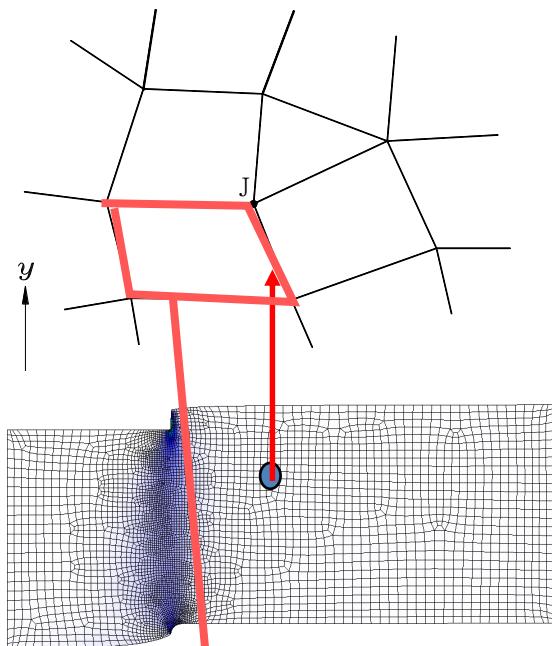
Extremize  $F(\phi)$

$$\text{subject to } \phi(0) = 0, \quad \phi(1) = 0$$

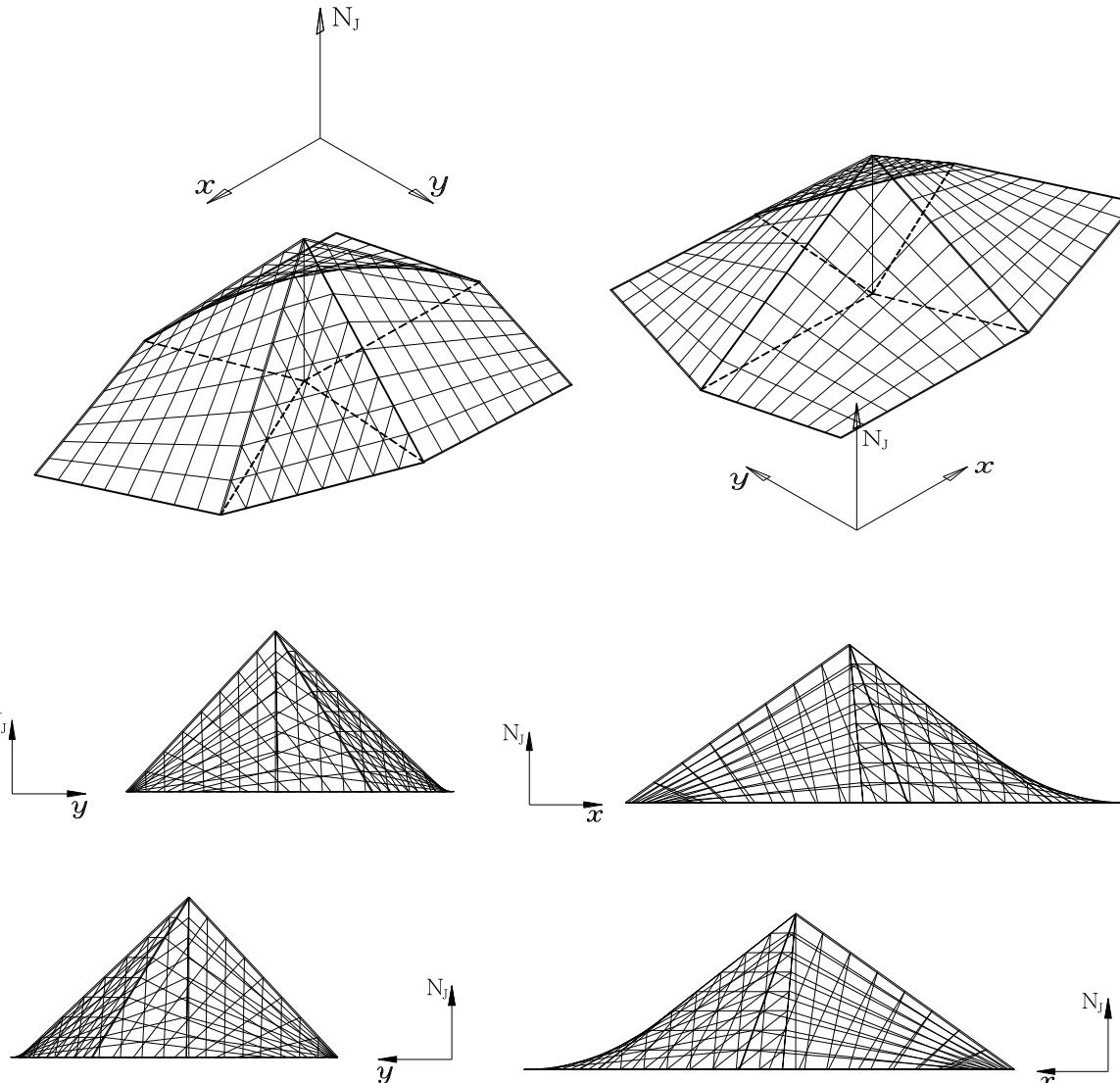
$$\tilde{\phi}(x, y) = \phi_1 N_1(x, y) + \phi_2 N_2(x, y) + \dots = \sum_J \phi_J N_J \quad N_J = ?, \quad \phi_J = ?$$

절점치

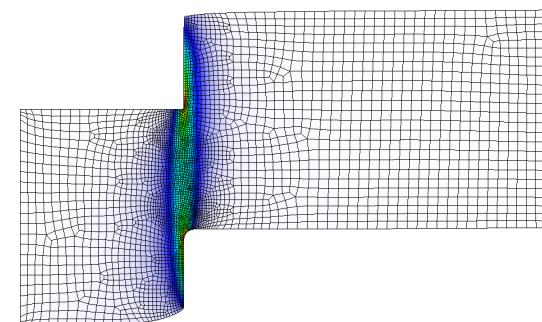
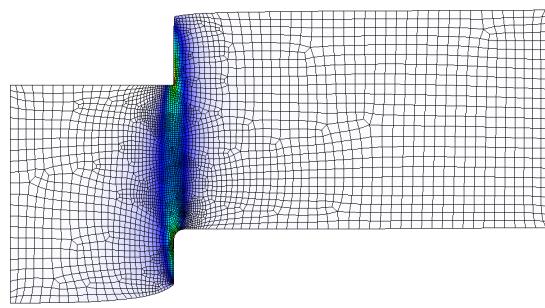
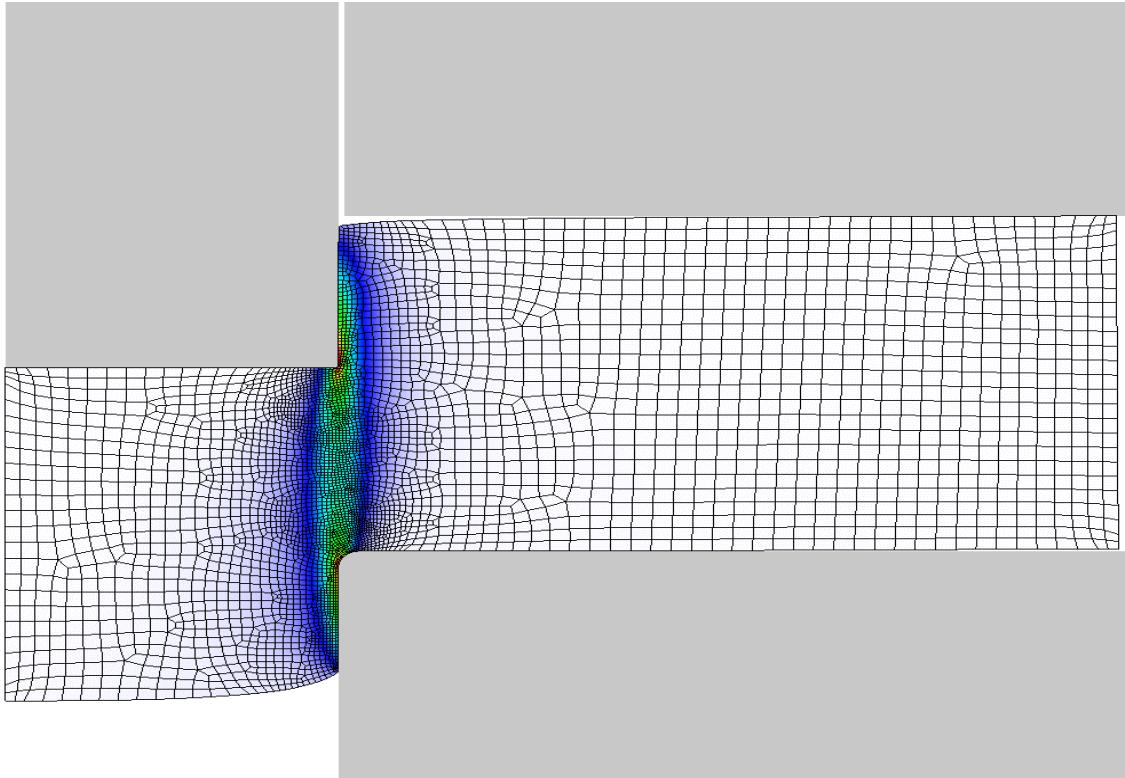
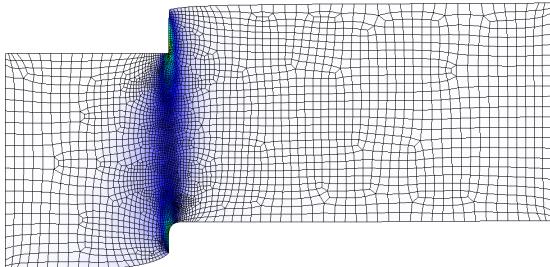
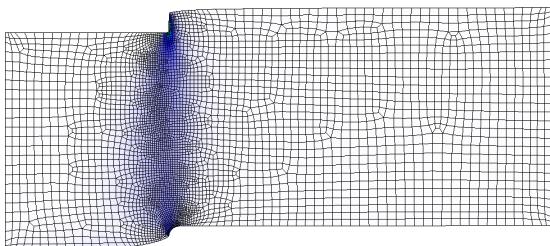
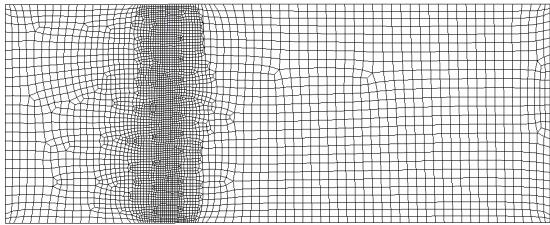
## 요소망



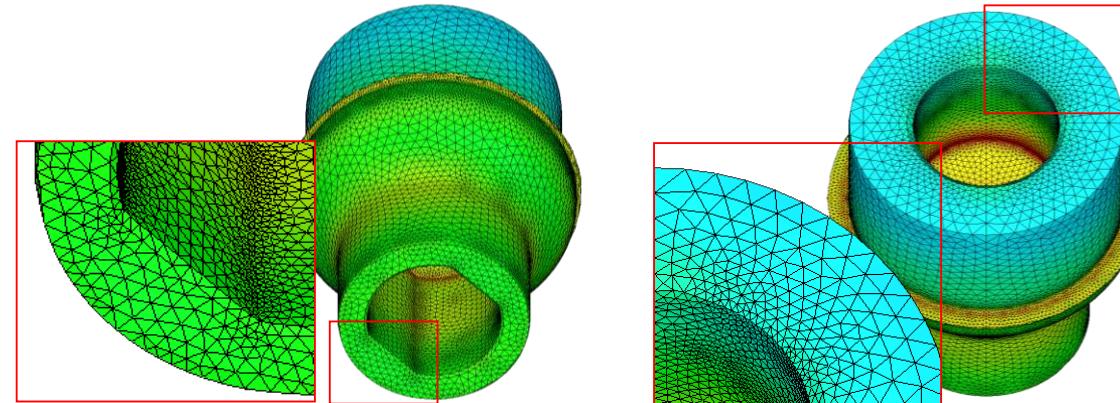
## 보간함수 $N_J(x, y)$



## ▣ 사각형요소의 지능적 재구성



## ▲ 육면체요소망의 지능적 재구성



**작은 요소 수로 형상을 정확하게!**

요소 수 많아지면 요소망재구성 많아져 수치적 순화가 커짐

