

III. Basis of Mechanics

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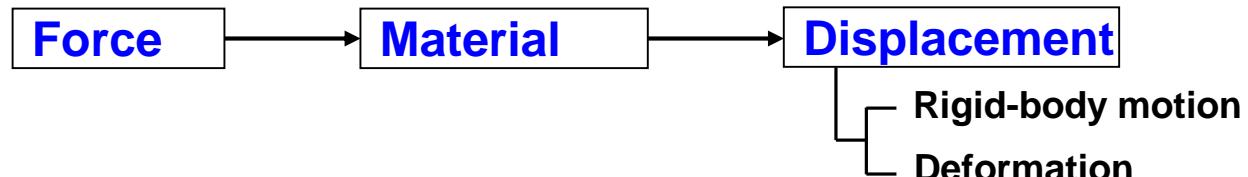


3.1 General Consideration



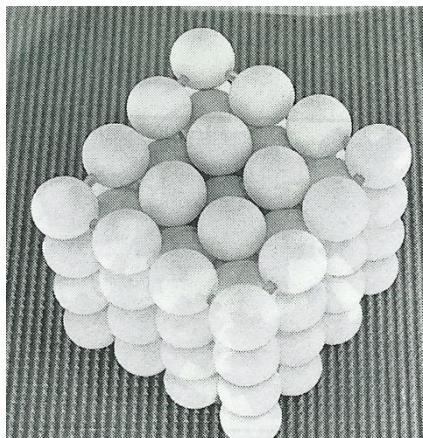
Basis of mechanics

- Three major factors

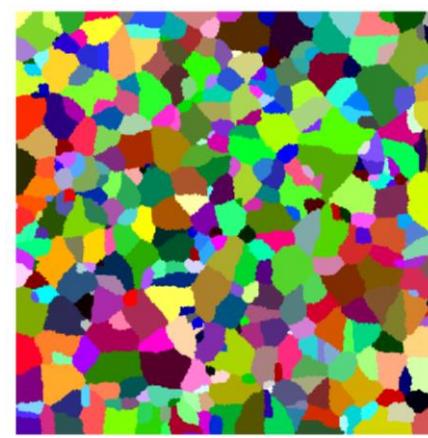


- Material and Continuum, Deformation of material

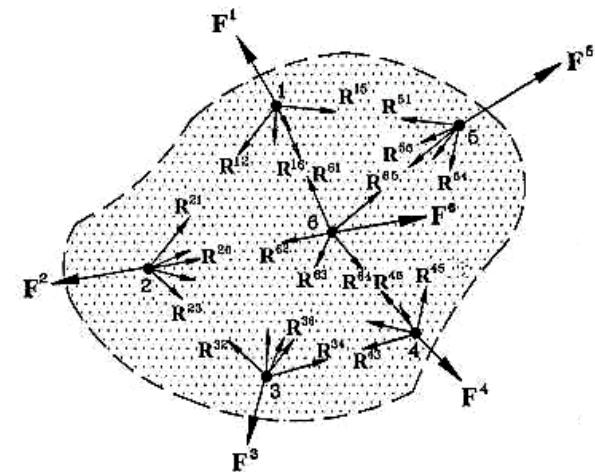
- Solid



<Grain>



<Grain and boundary>



<Continuum>

- Rigid-body : No change in distance between particles occurs under external force.
- Elastic deformation: Deformation due to external force disappearing when it removes.
- Plastic deformation: Deformation due to external force remaining when it removes.

- Fluid including gas



Some details of the 3 major factors

○ Force

Force	Internal	Action and reaction force	
	External	Traction	Exerted load, Reaction force
		Body force	Gravity, Magnetic force

○ Displacement or motion

- Rigid-body motion
- Deformation

○ Material: Set of particles continuously distributed → Continuum

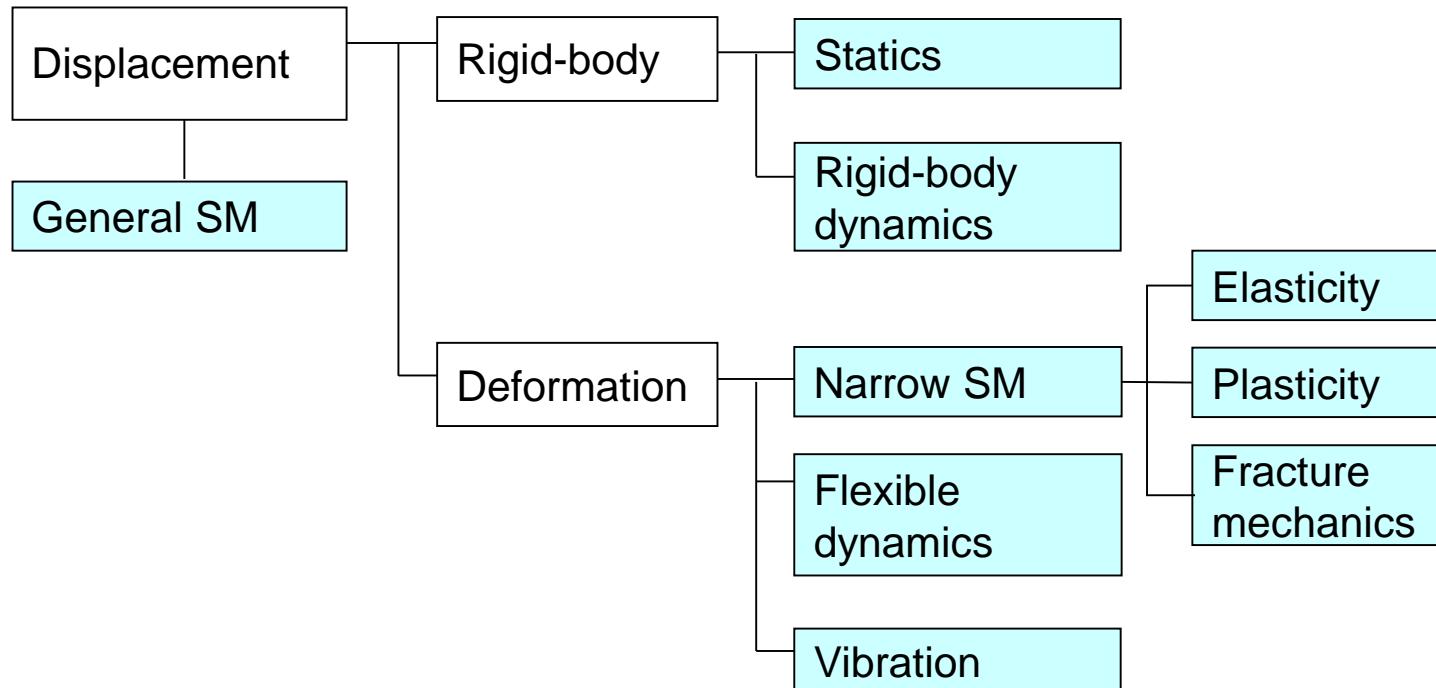
○ Solid

- Rigid-body: No relationship between force and displacement is needed.
- Elastically deformable body: Linear elastic(Hooke's law) and Non-elastic, Isotropic(Common material) and Anisotropic (Composite material)
- Plastic body: Yield criterion, Isotropic hardening, Kinematic hardening

○ Fluid including gas



Classification of solid mechanics (SM)





3.2 Newton's Law of Motion and Statics



Newton's law of motion

Newton's law

2nd
law

Sum of all the forces exerting on a particle (\mathbf{f}) is equal to the acceleration (\mathbf{a}) multiplied by the mass (m), i.e., $\mathbf{f} = m\mathbf{a}$.

3rd
law

Action and reaction law: Two internal forces exerting between two particles have the same magnitude and line of action and the opposite direction.

$$\mathbf{F}^i + \sum_j \mathbf{R}^{ij} = \mathbf{0}$$

$i, j = 1, 2, \dots, \infty$

$$\mathbf{R}^{ij} = -\mathbf{R}^{ji}$$

$$\sum_i (\mathbf{F}^i + \sum_j \mathbf{R}^{ij}) = \mathbf{0} \rightarrow \sum_i \mathbf{F}^i + \sum_i \sum_j \mathbf{R}^{ij} = \mathbf{0}$$

$$\sum_i \sum_j \mathbf{R}^{ij} = \mathbf{0}$$

$$\begin{cases} x-2=0 \\ y-3=0 \end{cases} \rightarrow \begin{cases} (O) \\ (X) \end{cases} \quad x+y-5=0$$

$$\sum \mathbf{F}^i = \mathbf{0}$$

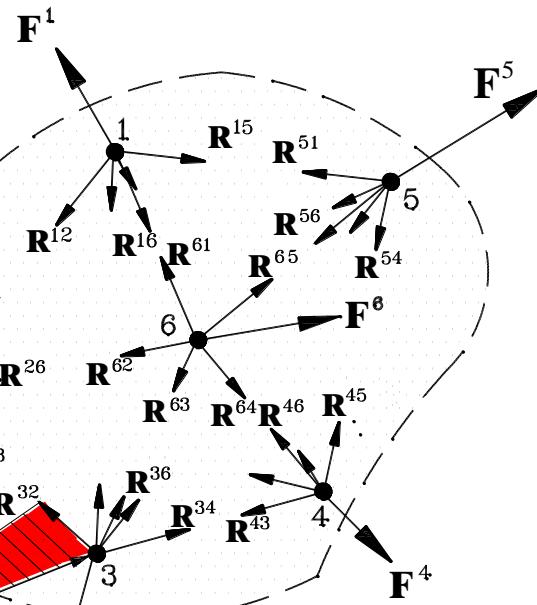
- A material is a set of infinite number of particles.

\mathbf{F}^i : Sum of external force exerting

on particle i

\mathbf{R}^{ij} : Internal force exerting on particle j from particle i

All forces are bounding vectors.



$$\mathbf{r}_2 \times \mathbf{R}^{23} + \mathbf{r}_3 \times \mathbf{R}^{32} = \mathbf{0}$$

$$\sum_i \mathbf{r}_i \times \left(\mathbf{F}^i + \sum_j \mathbf{R}^{ij} \right) = \mathbf{0} \rightarrow \sum_i \mathbf{r}_i \times \mathbf{F}^i + \sum_i \sum_j \mathbf{r}_i \times \mathbf{R}^{ij} = \mathbf{0}$$

$$\sum_i \sum_j \mathbf{r}_i \times \mathbf{R}^{ij} = \mathbf{0}$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^i = \mathbf{0}$$

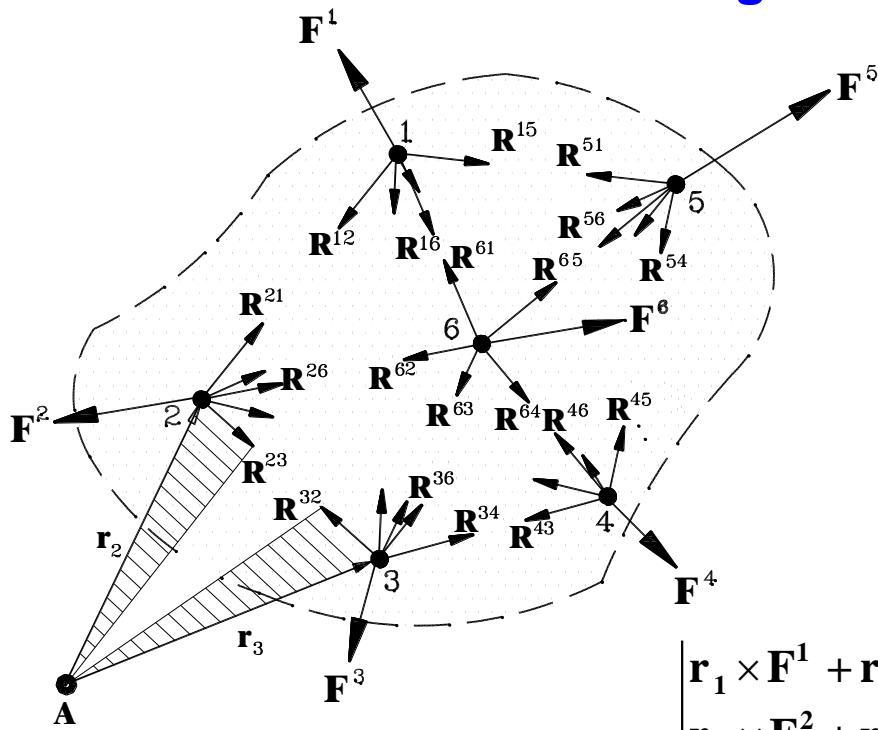
Requirement on equilibrium	$\sum \mathbf{F}^i = \mathbf{0}$ or $\sum \mathbf{F} = \mathbf{0}$
	$\sum_i \mathbf{r}_i \times \mathbf{F}^i = \mathbf{0}$ or $\sum \mathbf{M}_A = \mathbf{0}$

- The above requirement of equilibrium should satisfy for all subsystem as well as the whole system.

→ Leading to differential equations, for example, equations of equilibrium.



Derivation of requirement on equilibrium using a six-particle body



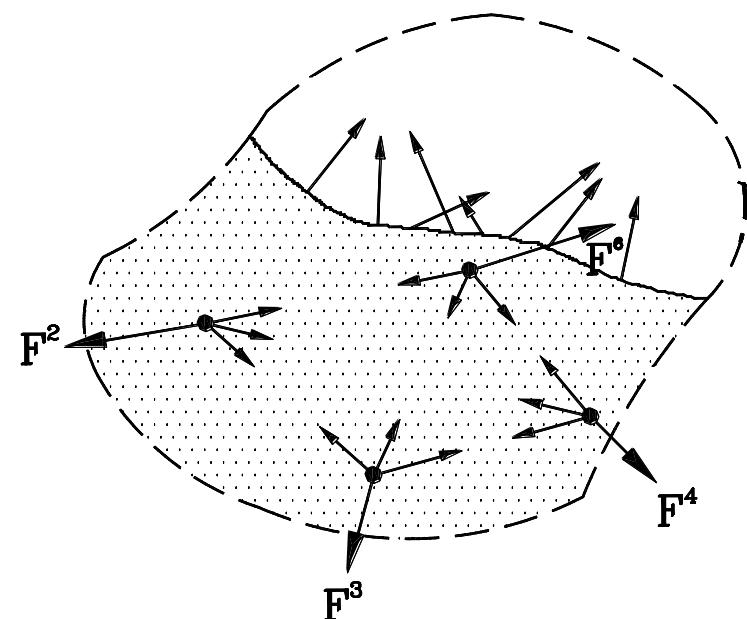
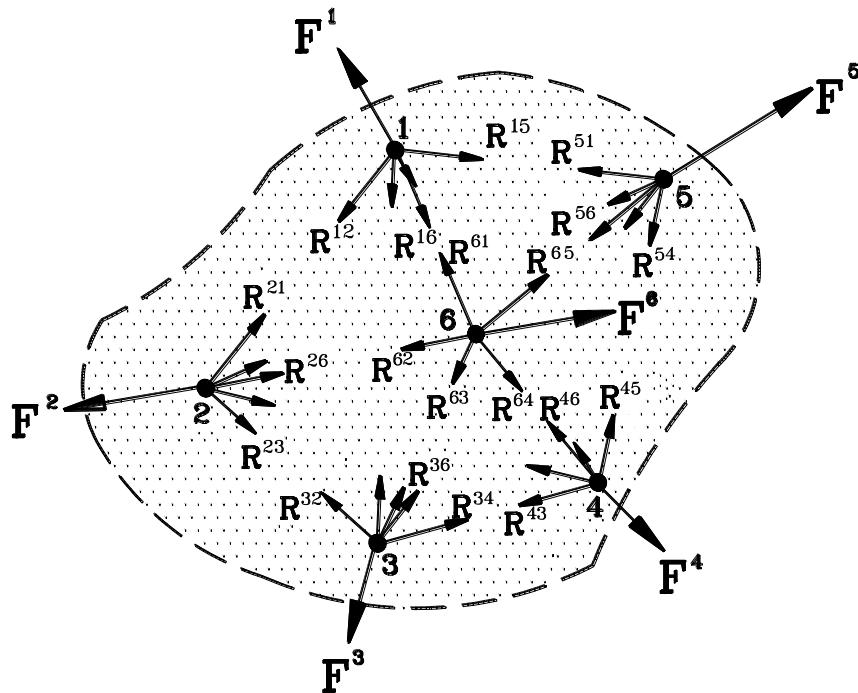
$$\begin{aligned}
 & \boxed{\begin{array}{l} \mathbf{F}^1 + \mathbf{R}^{12} + \mathbf{R}^{13} + \mathbf{R}^{14} + \mathbf{R}^{15} + \mathbf{R}^{16} = \mathbf{0} \\ \mathbf{F}^2 + \mathbf{R}^{21} + \mathbf{R}^{23} + \mathbf{R}^{24} + \mathbf{R}^{25} + \mathbf{R}^{26} = \mathbf{0} \\ \mathbf{F}^3 + \mathbf{R}^{31} + \mathbf{R}^{32} + \mathbf{R}^{34} + \mathbf{R}^{35} + \mathbf{R}^{36} = \mathbf{0} \\ \mathbf{F}^4 + \mathbf{R}^{41} + \mathbf{R}^{42} + \mathbf{R}^{43} + \mathbf{R}^{45} + \mathbf{R}^{46} = \mathbf{0} \\ \mathbf{F}^5 + \mathbf{R}^{51} + \mathbf{R}^{52} + \mathbf{R}^{53} + \mathbf{R}^{54} + \mathbf{R}^{56} = \mathbf{0} \\ \mathbf{F}^6 + \mathbf{R}^{61} + \mathbf{R}^{62} + \mathbf{R}^{63} + \mathbf{R}^{64} + \mathbf{R}^{65} = \mathbf{0} \end{array}} \\
 + & \quad \sum \mathbf{F}^i = \mathbf{0} \quad \therefore \mathbf{R}^{ij} = -\mathbf{R}^{ji}
 \end{aligned}$$

$$\begin{aligned}
 & \boxed{\begin{array}{l} \mathbf{r}_1 \times \mathbf{F}^1 + \mathbf{r}_1 \times \mathbf{R}^{12} + \mathbf{r}_1 \times \mathbf{R}^{13} + \mathbf{r}_1 \times \mathbf{R}^{14} + \mathbf{r}_1 \times \mathbf{R}^{15} + \mathbf{r}_1 \times \mathbf{R}^{16} = \mathbf{0} \\ \mathbf{r}_2 \times \mathbf{F}^2 + \mathbf{r}_2 \times \mathbf{R}^{21} + \mathbf{r}_2 \times \mathbf{R}^{23} + \mathbf{r}_2 \times \mathbf{R}^{24} + \mathbf{r}_2 \times \mathbf{R}^{25} + \mathbf{r}_2 \times \mathbf{R}^{26} = \mathbf{0} \\ \mathbf{r}_3 \times \mathbf{F}^3 + \mathbf{r}_3 \times \mathbf{R}^{31} + \mathbf{r}_3 \times \mathbf{R}^{32} + \mathbf{r}_3 \times \mathbf{R}^{34} + \mathbf{r}_3 \times \mathbf{R}^{35} + \mathbf{r}_3 \times \mathbf{R}^{36} = \mathbf{0} \\ \mathbf{r}_4 \times \mathbf{F}^4 + \mathbf{r}_4 \times \mathbf{R}^{41} + \mathbf{r}_4 \times \mathbf{R}^{42} + \mathbf{r}_4 \times \mathbf{R}^{43} + \mathbf{r}_4 \times \mathbf{R}^{45} + \mathbf{r}_4 \times \mathbf{R}^{46} = \mathbf{0} \\ \mathbf{r}_5 \times \mathbf{F}^5 + \mathbf{r}_5 \times \mathbf{R}^{51} + \mathbf{r}_5 \times \mathbf{R}^{52} + \mathbf{r}_5 \times \mathbf{R}^{53} + \mathbf{r}_5 \times \mathbf{R}^{54} + \mathbf{r}_5 \times \mathbf{R}^{56} = \mathbf{0} \\ \mathbf{r}_6 \times \mathbf{F}^6 + \mathbf{r}_6 \times \mathbf{R}^{61} + \mathbf{r}_6 \times \mathbf{R}^{62} + \mathbf{r}_6 \times \mathbf{R}^{63} + \mathbf{r}_6 \times \mathbf{R}^{64} + \mathbf{r}_6 \times \mathbf{R}^{65} = \mathbf{0} \end{array}} \\
 + & \quad \sum \mathbf{r}_i \times \mathbf{F}^i = \mathbf{0} \quad \therefore \mathbf{r}_i \times \mathbf{R}^{ij} = -\mathbf{r}_j \times \mathbf{R}^{ij}
 \end{aligned}$$

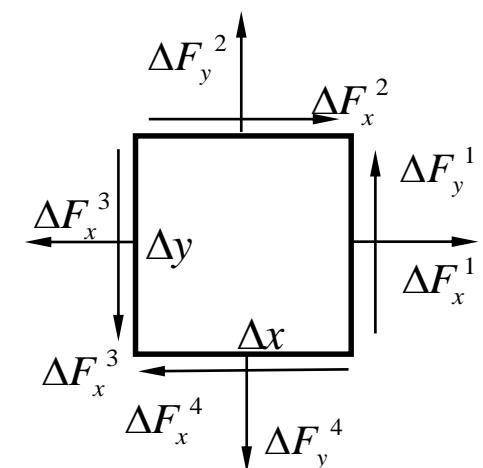
*Actual number of particles is infinite.



Subsystem



* The requirement of equilibrium should satisfy for any subsystem. This statement is the same with the following statement: The requirement of equilibrium should satisfy for arbitrary infinitesimal area in 2D or volume in 3D, leading to differential equation. Note that we should define stress, i.e., force per unit area to define the force exerting on boundary of the infinitesimal area in 2D or volume in 3D.

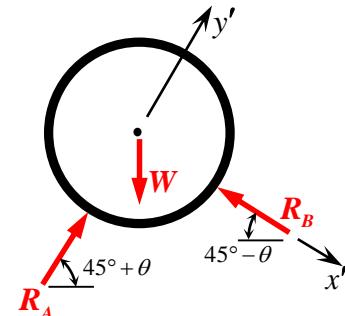
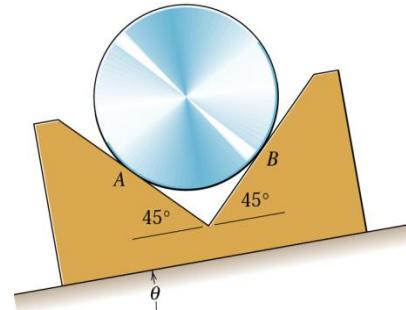
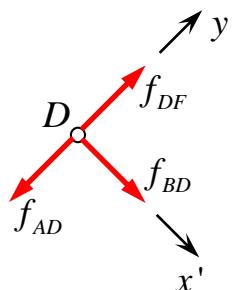
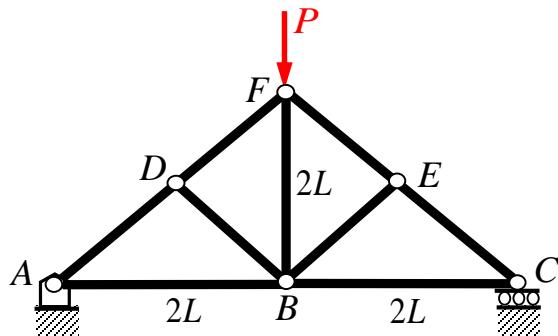




Requirement on equilibrium in 2D and 3D

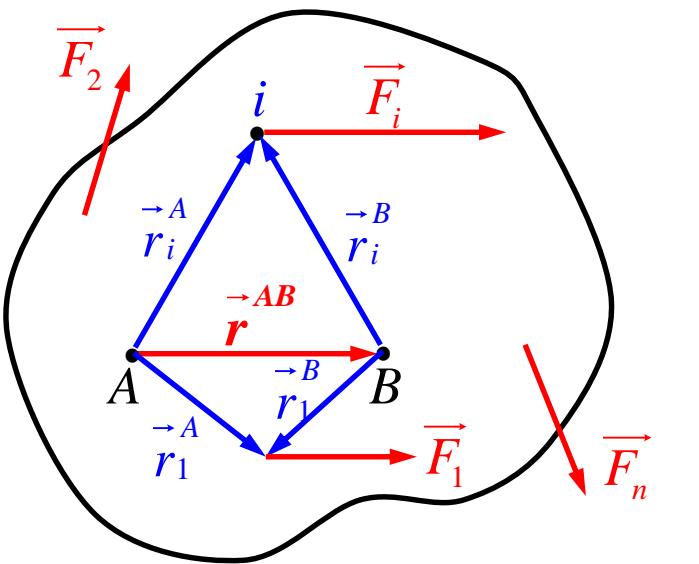
2D plane	3D space	Tip
$\sum \vec{F} = 0 \rightarrow \left(\begin{array}{l} \sum F_x = 0 \\ \sum F_y = 0 \end{array} \right)$	$\sum \vec{F} = 0 \rightarrow \left(\begin{array}{l} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{array} \right)$	$x, y,$ and z can be replaced by any three independent directions
$\sum \vec{M}_A = 0 \rightarrow (\sum M_{A,z} = 0)$	$\sum \vec{M}_A = 0 \rightarrow \left(\begin{array}{l} \sum M_{A,x} = 0 \\ \sum M_{A,y} = 0 \\ \sum M_{A,z} = 0 \end{array} \right)$	
3 equations	6 equations	

◎ Examples





Caution in applying requirements on equilibrium



\vec{r}_i^A and \vec{r}_i^B are dependent on the position of point i , while \vec{r}^{AB} is fixed.

$$\left(\begin{array}{l} \sum \vec{F}_i = \vec{0} \\ \sum \vec{M}_A = \sum_i \vec{r}_i^A \times \vec{F}_i = \vec{0} \end{array} \right) \rightarrow \sum \vec{M}_B = ?$$

$$\begin{aligned}
 \sum \vec{M}_B &= \sum_i \vec{r}_i^B \times \vec{F}_i \\
 &= \sum_i (\vec{r}_i^A - \vec{r}^{AB}) \times \vec{F}_i \\
 &= \sum_i \vec{r}_i^A \times \vec{F}_i - \sum_i \vec{r}^{AB} \times \vec{F}_i \\
 &= \sum \vec{M}_A - \vec{r}^{AB} \times \sum \vec{F}_i \\
 &= \vec{0}
 \end{aligned}$$

$\therefore \sum \vec{F} = \vec{0}, \sum \vec{M}_A = \vec{0} \rightarrow \sum \vec{M}_B = \vec{0}$

$\sum \vec{M}_A = \vec{0}, \sum \vec{M}_B = \vec{0} \rightarrow \sum \vec{F} = \vec{0}$

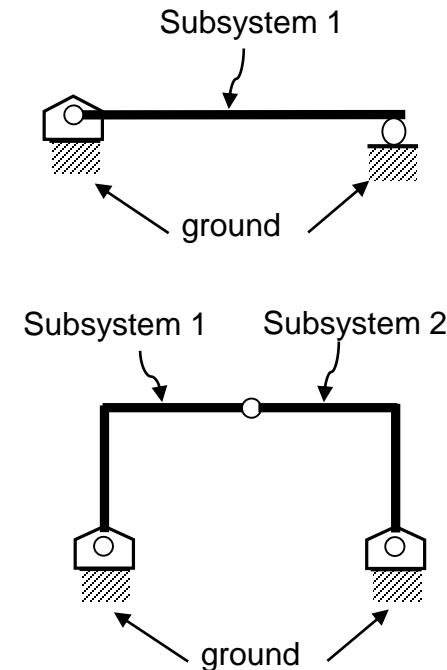
(If $\vec{r}^{AB} \nparallel \sum \vec{F}$)



Statics

◎ Statics and solid mechanics with three factors of mechanics

Factor	Statics	Solid mechanics
Force	Requirement on equilibrium	Equation of equilibrium or motion
Deformation	Rigid-body	Displacement-strain relation, Geometric compatibility
Material	Rigid-body	Elastic body: Hooke's law Plastic body: Plastic flow rule
BC	Geometric(essential) BC	Geometric(essential) BC, mechanical(natural) BC



◎ Entire system

- Ground and support
- Subsystem

◎ Separation of subsystems

- Essential separation of subsystems from support or ground
- Division of subsystems if needed
 - When number of equations should be greater than that of new knowns
 - When internal resultant forces is to be seen

◎ Newtonian mechanics
based on vector quantities

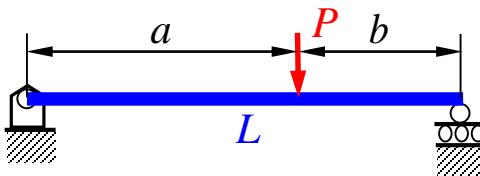


Analytical mechanics
based on energy

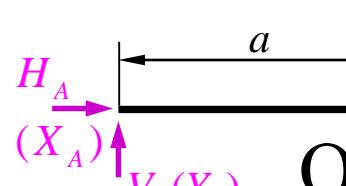


Free-body diagram and application of RE on it

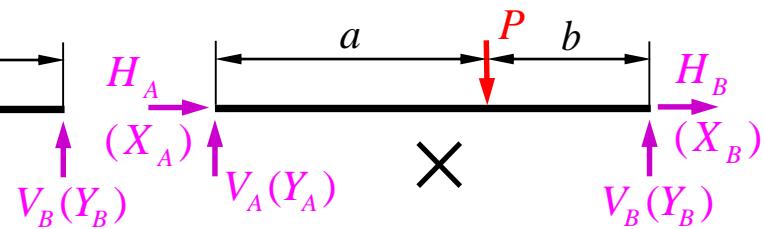
◎ Simply supported beam



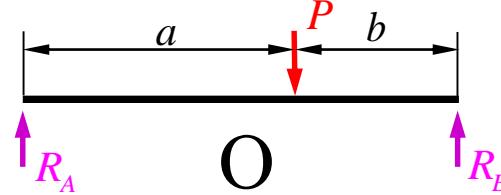
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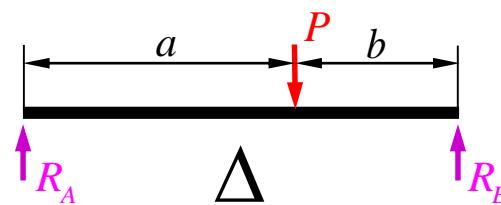
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○ Requirement on equilibrium

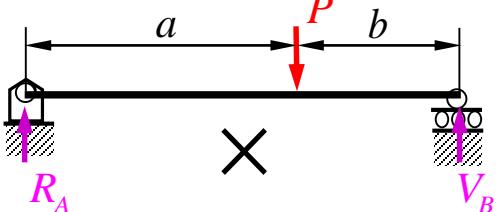
$$\text{No. 1} \begin{cases} \sum F_y = 0 ; R_A + R_B - P = 0 \\ \sum F_x = 0 ; 0 = 0 \\ \sum M_A = 0 ; LR_B - aP = 0 \end{cases} \rightarrow \begin{cases} R_A = (b/L)P \\ R_B = (a/L)P \end{cases}$$

< F.B.D. No. 2 >



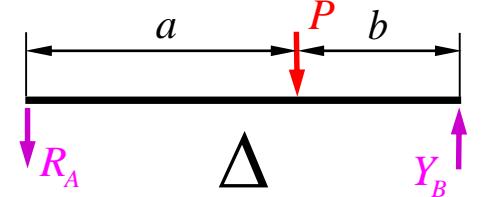
$$\text{No. 2} \begin{cases} \sum M_A = 0 ; LR_B - aP = 0 \\ \sum F_x = 0 ; 0 = 0 \\ \sum M_B = 0 ; -LR_A + bP = 0 \end{cases} \rightarrow \begin{cases} R_A = (b/L)P \\ R_B = (a/L)P \end{cases}$$

< F.B.D. No. 3 >



$$\text{No. 3} \begin{cases} \sum F_y = 0 ; V_A + V_B - P = 0 \\ \sum F_x = 0 ; H_A = 0 \\ \sum M_A = 0 ; LV_B - aP = 0 \end{cases} \rightarrow \begin{cases} H_A = 0 \\ V_A = (b/L)P \\ V_B = (a/L)P \end{cases}$$

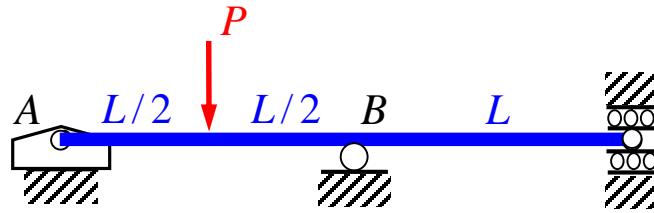
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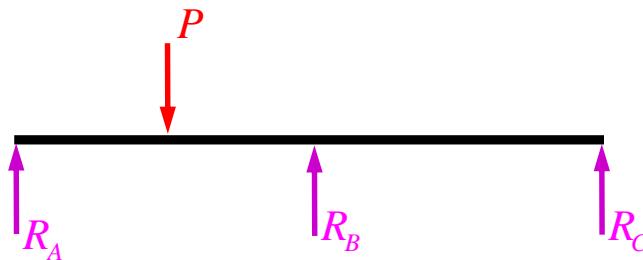


Statically indeterminate system

- ◎ 3 points supporting beam



< F.B.D. >



- Requirement on equilibrium

$$\sum F_x = 0 ; \quad 0 = 0 \quad (1)$$

$$\sum F_y = 0 ; \quad R_A + R_B + R_C - P = 0 \quad (2)$$

$$\sum M_A = 0 ; \quad LR_B + 2LR_C - (L/2)P = 0 \quad (3)$$

$$\sum M_B = 0 ; \quad -LR_A + LR_C + (L/2)P = 0 \quad (4)$$

$$\sum M_C = 0 ; \quad -2LR_A - LR_B + (3/2)LP = 0 \quad (5)$$

- Eqs. ① ~ ⑤ have only two independent equations.

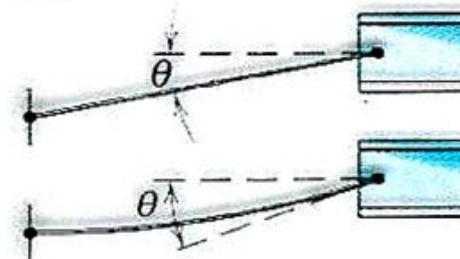
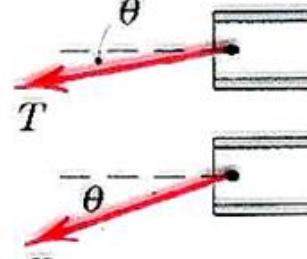
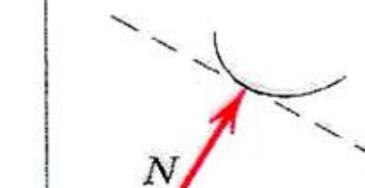
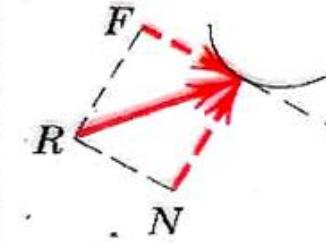
- Infinite number of equations can be made applying moment balance requirement at arbitrary point.

- Number of unknowns > number of equations
⇒ Statically indeterminate system

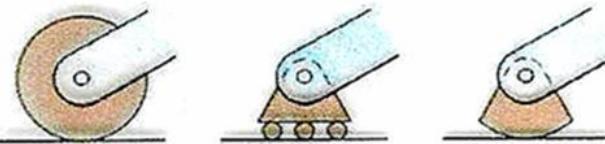
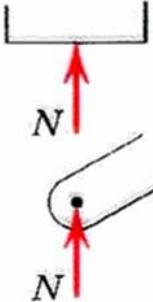
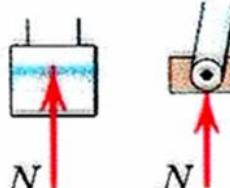
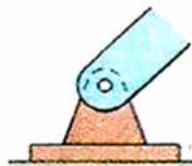
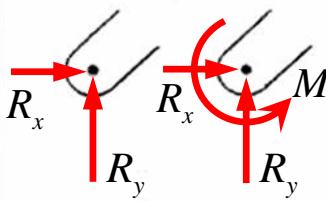
- Reactions in a statically indeterminate system can be determined not by statics but by solid mechanics.



Division of total system into subsystems

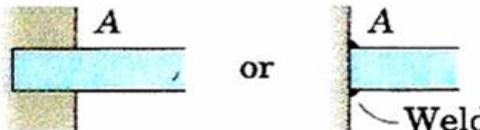
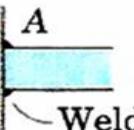
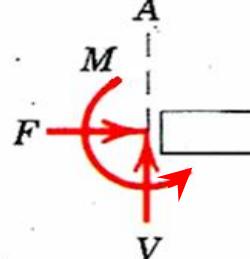
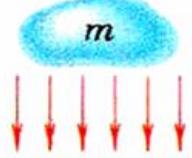
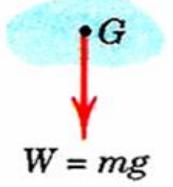
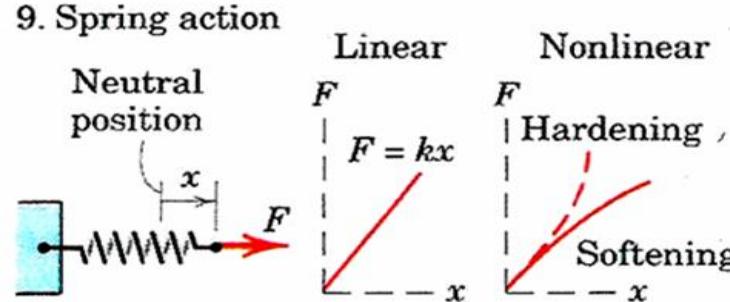
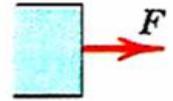
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to be Isolated
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible	  <p>Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.</p>
2. Smooth surfaces	 <p>Contact force is compressive and is normal to the surface.</p>
3. Rough surfaces	 <p>Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.</p>

Division of total system into subsystems

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (cont.)					
Type of Contact and Force Origin	Action on Body to be Isolated				
4. Roller support	 	 <p>Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.</p>			
5. Freely sliding guide		 <p>Collar or slider free to move along smooth guides; can support force normal to guide only.</p>			
6. Pin connection		<table border="0"> <tr> <td style="vertical-align: top;"> Pin free to turn </td> <td style="vertical-align: top;"> Pin not free to turn </td> <td style="vertical-align: top;"> A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components R_x and R_y. A pin not free to turn may also support a couple M. </td> </tr> </table> 	Pin free to turn	Pin not free to turn	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components R_x and R_y . A pin not free to turn may also support a couple M .
Pin free to turn	Pin not free to turn	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components R_x and R_y . A pin not free to turn may also support a couple M .			

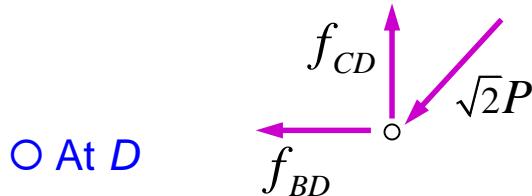
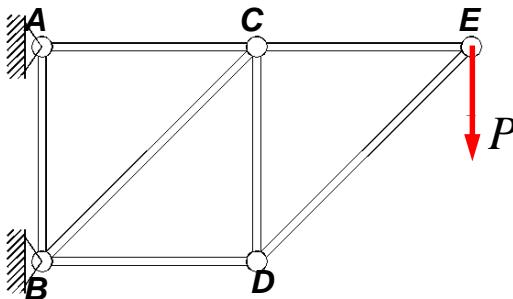


Division of total system into subsystems

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (cont.)	
Type of Contact and Force Origin	Action on Body to be Isolated
7. Built-in or fixed support  or 	 <p>A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.</p>
8. Gravitational attraction 	 <p>The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.</p>
9. Spring action  <p>Neutral position Linear: $F = kx$ Nonlinear: Hardening, Softening</p>	 <p>Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.</p>



Forces in members of a truss structure



$$\sum F_x = 0; -f_{BD} - \sqrt{2}P \times \frac{\sqrt{2}}{2} = 0$$

$$\sum F_y = 0; f_{CD} - \sqrt{2}P \times \frac{\sqrt{2}}{2} = 0$$

$$\therefore f_{BD} = -P, f_{CD} = P$$

Ans. $f_{AC} = 2P$ (in tension)

$f_{BC} = -\sqrt{2}P$ (in compression),

○ At E

$$\begin{aligned} \sum F_x &= 0; -f_{CE} - \frac{\sqrt{2}}{2} f_{DE} = 0 \\ \sum F_y &= 0; -P - \frac{\sqrt{2}}{2} f_{DE} = 0 \\ \therefore f_{DE} &= -\sqrt{2}P, f_{CE} = P \end{aligned}$$

○ At C

$$\sum F_x = 0; P - f_{AC} - \frac{\sqrt{2}}{2} f_{BC} = 0$$

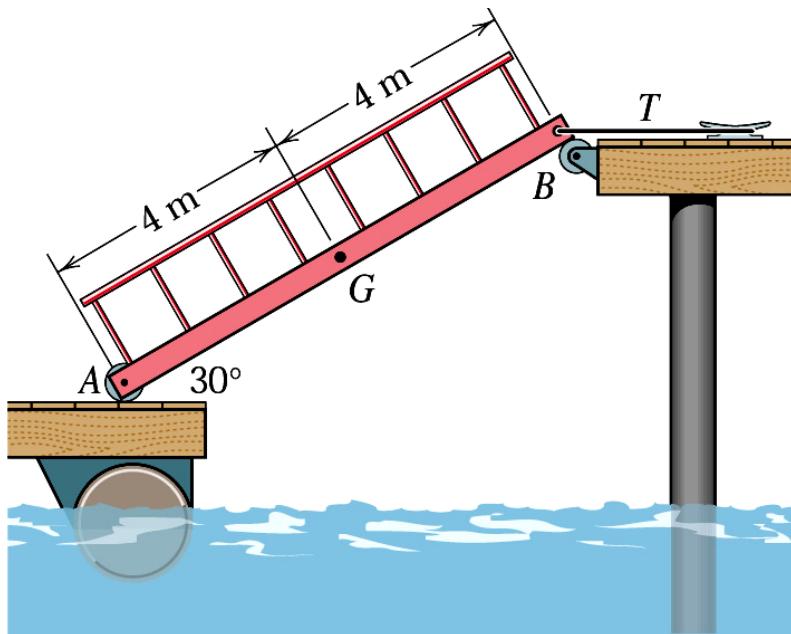
$$\sum F_y = 0; -P - \frac{\sqrt{2}}{2} f_{BC} = 0$$

$$\therefore f_{BC} = -\sqrt{2}P, f_{AC} = 2P$$

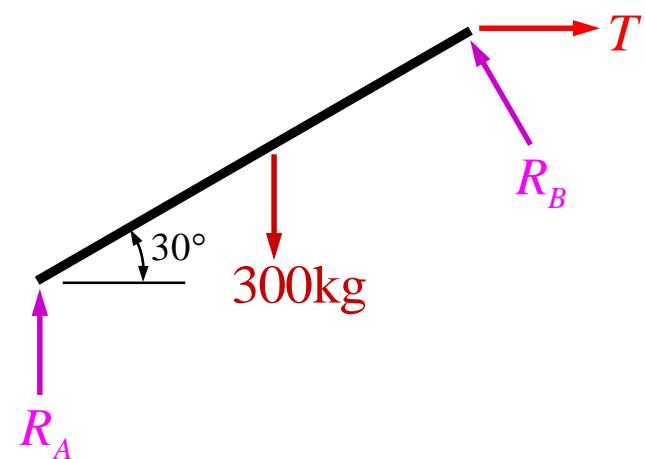


2D static problem

- Tension of cable, T ?



< F.B.D. >



- Applying RE on FBD

$$\sum F_x = 0 ; \quad T - R_B \sin 30^\circ = 0$$

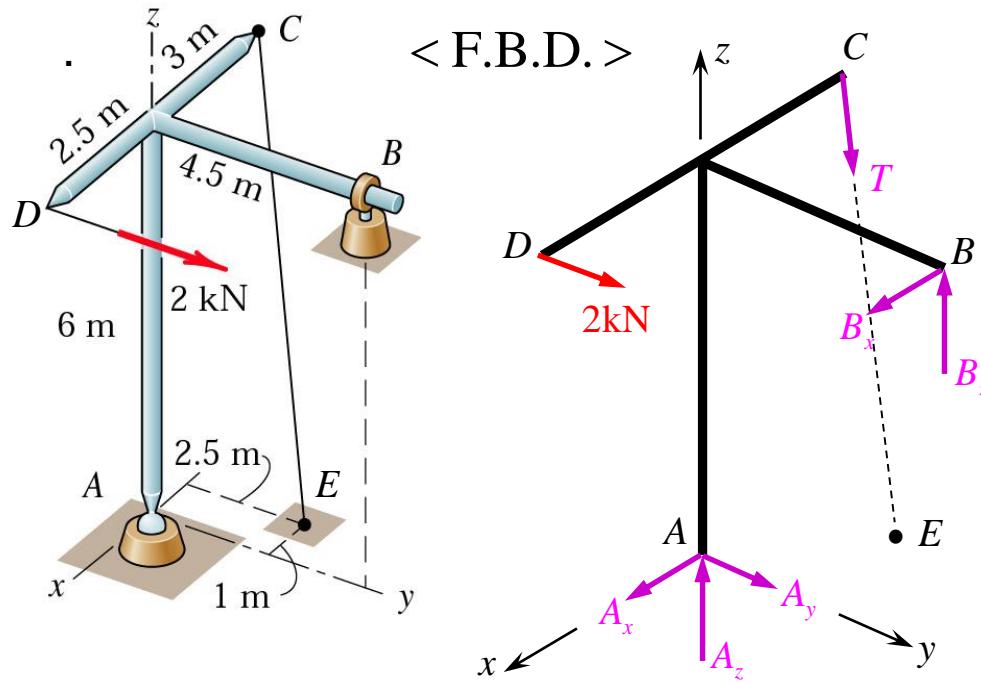
$$\sum F_y = 0 ; \quad R_A + R_B \cos 30^\circ - 300 = 0$$

$$\sum M_B = 0 ; \quad -8 \cos 30^\circ R_A + 4 \cos 30^\circ \times 300 = 0$$



3D static problem

○ Cable tension T ?



○ Vector expression of line segments CA and CE

$$\vec{r}_{CA} = -3\vec{i} - 6\vec{k}, \vec{r}_{CD} = 5.5\vec{i}, \vec{r}_{CB} = 3\vec{i} - 4.5\vec{k}$$

$$\vec{CE} = (-\vec{i} + 2.5\vec{j}) - (-3\vec{i} + 6\vec{k}) = 2\vec{i} + 2.5\vec{j} - 6\vec{k}$$

$$\frac{\vec{CE}}{|\vec{CE}|} \equiv \vec{ce} = (2\vec{i} + 2.5\vec{j} - 6\vec{k}) / \sqrt{46.2}$$

○ Requirement on equilibrium

$$\sum \vec{F} = 0; A_x + B_x + 2T / \sqrt{46.2} = 0$$

$$A_y + 2 + 2.5T / \sqrt{46.2} = 0$$

$$A_z + B_z + 2T / \sqrt{46.2} = 0$$

$$\begin{aligned} \sum \vec{M}_C = 0; & \vec{r}_{CA} \times (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \\ & + \vec{r}_{CD} \times (2\vec{j}) \\ & + \vec{r}_{CB} \times (B_x \vec{i} + B_z \vec{k}) = \vec{0} \end{aligned}$$

3.3 Unit and Dimension



Unit and dimension

◎ System of Units

- ※ Three basis units characterize the system of units.
- Conventional unit: Basis units are length, weight, time, current, temperature, etc.
- Science unit: Basis units are length, mass, time, current, temperature, etc.

◎ Dimension

- [Length] \equiv [L], [Mass] \equiv [M], [Time] \equiv [T], [Weight] \equiv [F]
- Basis units of system of science units: [L], [M], [T]
- Basis units of system of conventional units : [L], [F], [T]



British unit and SI unit

◎ British unit

- Basis unit: ft, lb, sec
- British unit belongs to conventional unit, i.e., lb means basically lb_f (pound force).
- Unit for mass : slug = $\text{lb} \cdot \text{sec}^2/\text{ft}$

◎ SI unit (The International System of Units)

- Basis unit: m, kg, s
- In SI unit, kg means basically kg_m (kilogram mass).
- Unit for force: $N = \text{kg} \cdot \text{m/s}^2$

◎ Entangled or mixed use of conventional and science units

- Many countries belonging to the Commonwealth of Nations are using British unit.
- The other countries are using SI unit. However, most people are using the SI unit like British unit, i.e., they are using kg (i.e., kg force) instead of N (Newton).
- Sometimes mass is expressed in $\text{kg} \cdot \text{s}^2/\text{m}$, where kg means kg_f .



Basis unit, complement unit, assembled unit

◎ Basis unit

- Length : m, ft
- Mass : kg, slug
- Time : s, sec
- Current : A
- Temperature : K
- Voltage : V

◎ Complement unit

- rad

◎ Assembled unit

- Velocity : m/s, ft/sec
- $1N = 1kg \cdot m/s^2$, lb, kg
- Pressure : $1Pa = 1N/m^2$, psi = lb/in², psf = lb/ft², kg/m², kg/mm²
- Work : $1J = 1N \cdot m$, lb · ft, kg · m
- Power : $1W = 1J/s$, lb · ft/sec, kg · m/s, PS, hp
- Moment : N · m, lb · ft, kg · m

◎ Factors for unit

- T(10^{12}), G(10^9), M(10^6), k(10^3), c(10^{-2}), m(10^{-3}), μ (10^{-6}), n(10^{-9}), p(10^{-12})

◎ Relationship between the two systems of unit

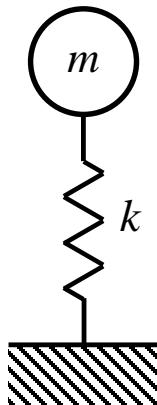
- 1ft = 0.3048 m, 1 in = 25.4 mm, 1lb = 0.4536 kg

All units should be expressed in Roman characters while variables in Italic or bold Roman characters.



Example 1

○



$$\omega_n = \sqrt{\frac{k}{m}}, \quad k = 1\text{kg/mm}, \quad m = 1\text{kg}$$

$$\begin{aligned}\omega_n &= \sqrt{\frac{1\text{kg}_f/\text{mm}}{1\text{kg}_m}} = \sqrt{\frac{1\text{kg}_m g}{1\text{kg}_m \text{mm}}} = \sqrt{\frac{1\text{kg}_m \times 10^4 \text{ mm}}{1\text{kg}_m \cancel{\text{mm}} \cdot \text{s}^2}} \\ &= 100 \frac{1}{\text{s}} \\ \omega_n &= \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{50}{\pi} \text{ (Hz)}\end{aligned}$$

○ Mistake due to gravitational acceleration.

Calculation of blow energy per stroke in hammer forging

$$E = (m \times g + p \times A)H$$

$$\begin{aligned}E &= (56500\text{kg} \times 9.8 \text{ m/s}^2 + 7\text{kg/cm}^2 \times 17671.5\text{cm}^2)0.75\text{m} \\ &= (553700\text{kg} + 123700\text{kg})0.75\end{aligned}$$

$$E = 508,050\text{kg.m} \rightarrow 4,982,250 \text{ N-m}$$

$$\begin{aligned}m &= 56,500\text{kg} \\ g &= 9.8 \text{ m/s}^2 \\ p &= 7\text{kg/cm}^2 \\ A &= 17671.5 \text{ cm}^2 \\ H &= 0.75\text{m}\end{aligned}$$



From a lecture note floating in internet

Physical quantities and dimensions	SI	U.S customary
mass [M]	kg	slug
length [L]	m	ft
time [T]	s	sec
force [F]	N	lb

$$\cancel{F = ma} \Rightarrow \cancel{N = kg \cdot m / sec^2}$$

Original

$$W(\text{weight}) = m(\cancel{kg}) \times g(\cancel{m / sec^2}), \quad g = 9.806 \cancel{m / sec^2}$$

$$F = ma \Rightarrow N = kg \cdot m / s^2$$

Modified

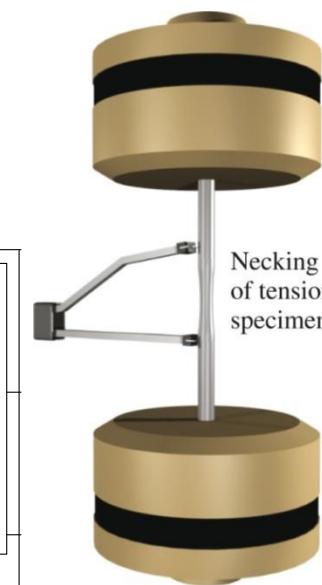
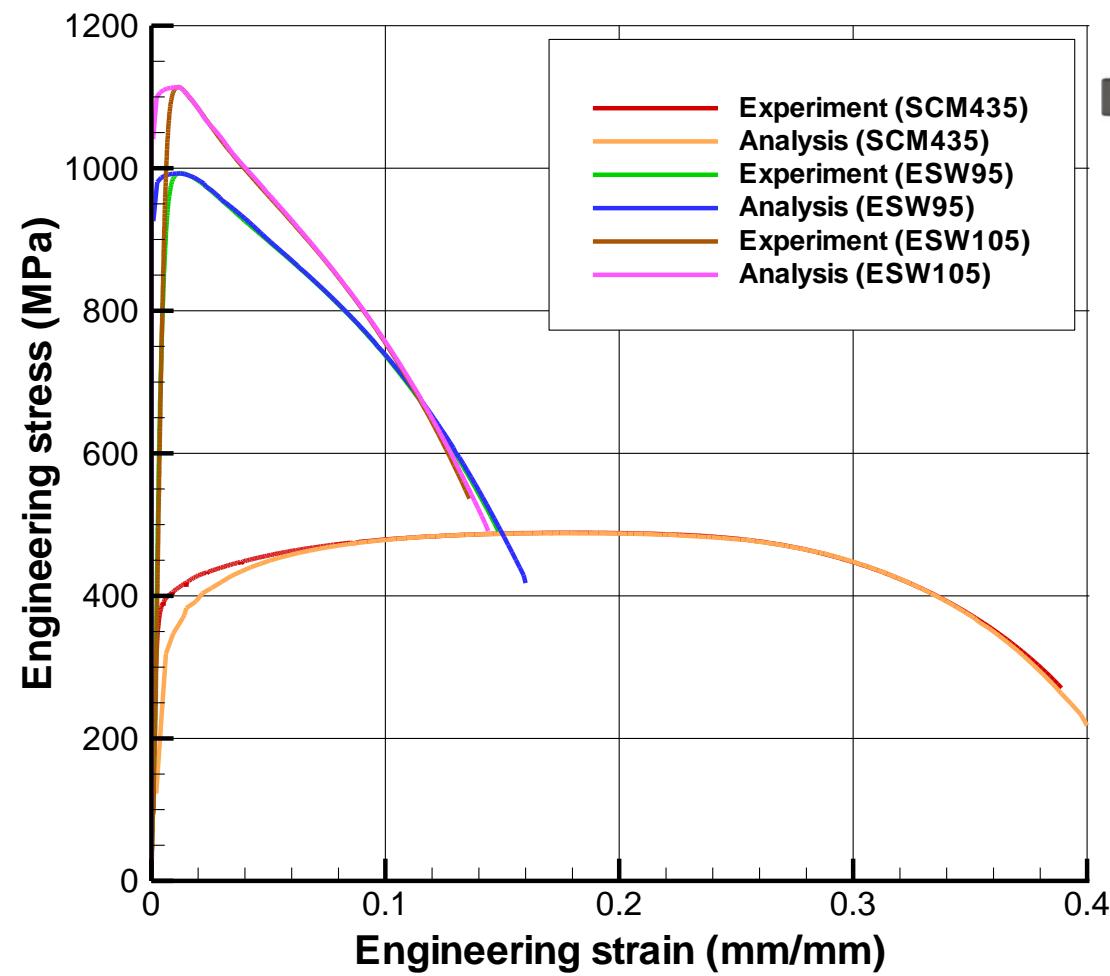
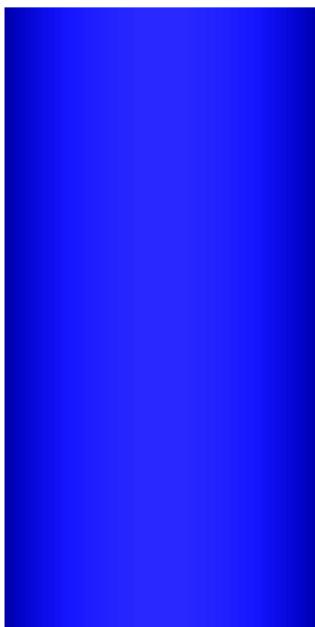
$$W(\text{weight}) = m(kg) \times g(m/s^2), \quad g = 9.806 m/s^2$$



3.4 Material



Tensile test

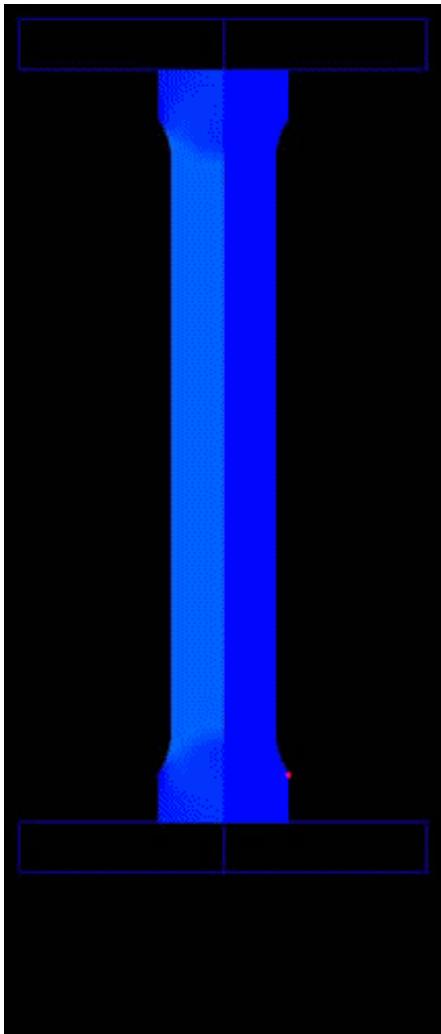


Necking
of tension
specimen

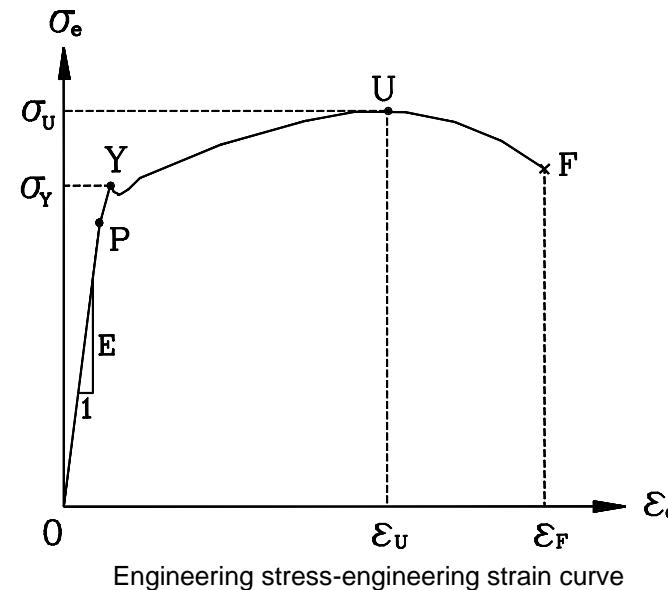
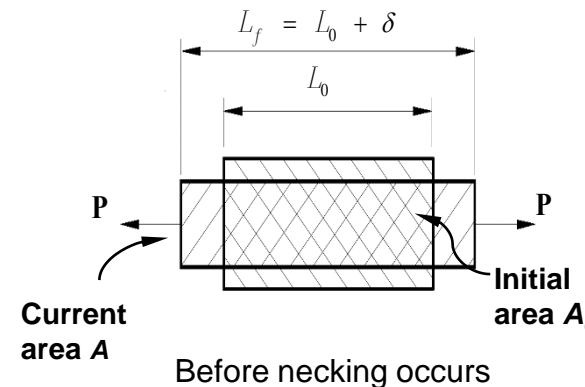


Tensile test

Tensile test



Predictions of tensile test



σ_y : Yield strength
 σ_u : Tensile strength
 $\varepsilon_{max} \times 100(%)$: Elongation

Definition of ε_e and σ_e

$$\bullet \varepsilon_e = \frac{\delta}{L_0}, \quad \sigma_e = \frac{P}{A_0} \quad e: \text{engineering}$$

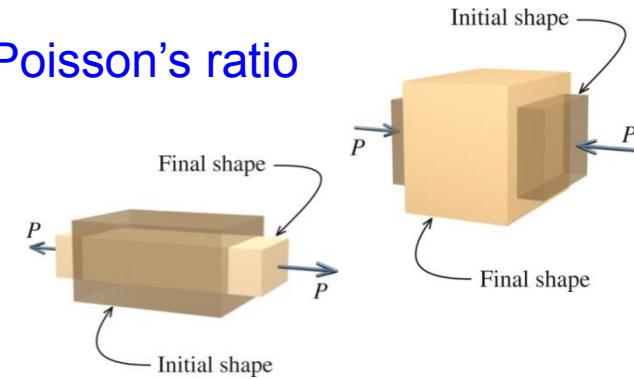
↑ ↑

Engineering strain, Engineering stress
(Engineering = Conventional=Nominal)

Hooke's law in uniaxial loading

- $\sigma_t = E\varepsilon_t, \sigma = E\varepsilon$
- $\sigma_x = E\varepsilon_x, \sigma_{xx} = E\varepsilon_{xx}$

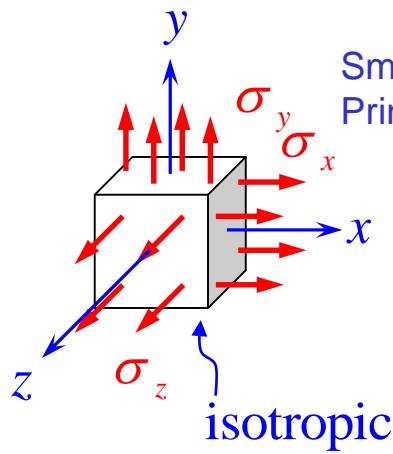
Poisson's ratio



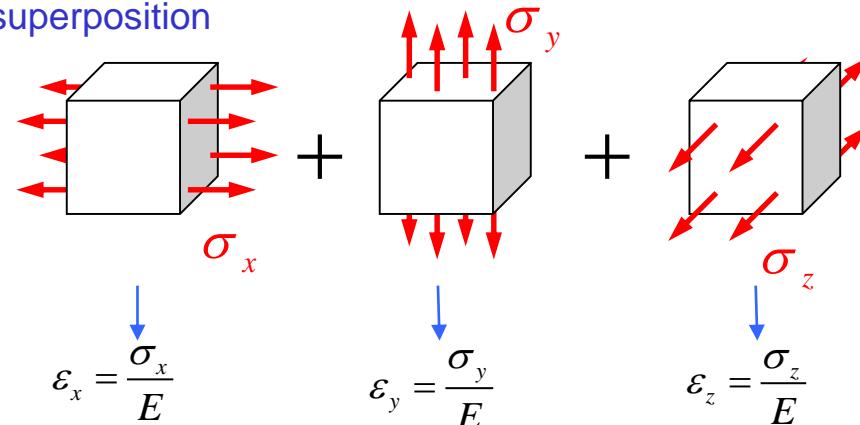
$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}} = -\frac{\varepsilon_t}{\varepsilon_a}$$



Generalized Hooke's law for an isotropic material

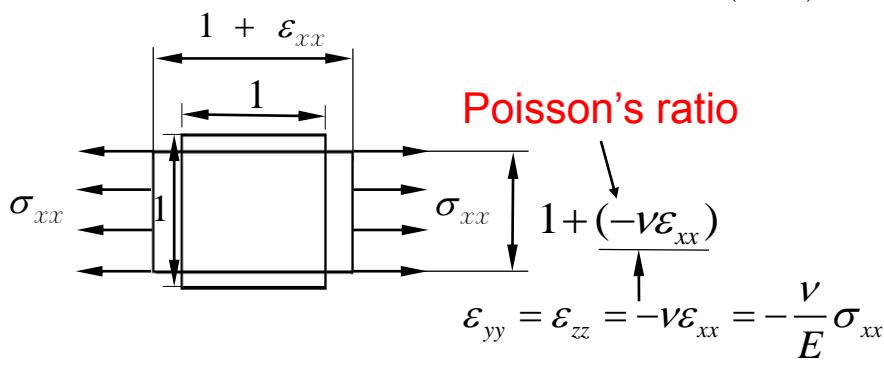


Small deformation
Principle of superposition



$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$$

$$G = \frac{E}{2(1+\nu)}$$



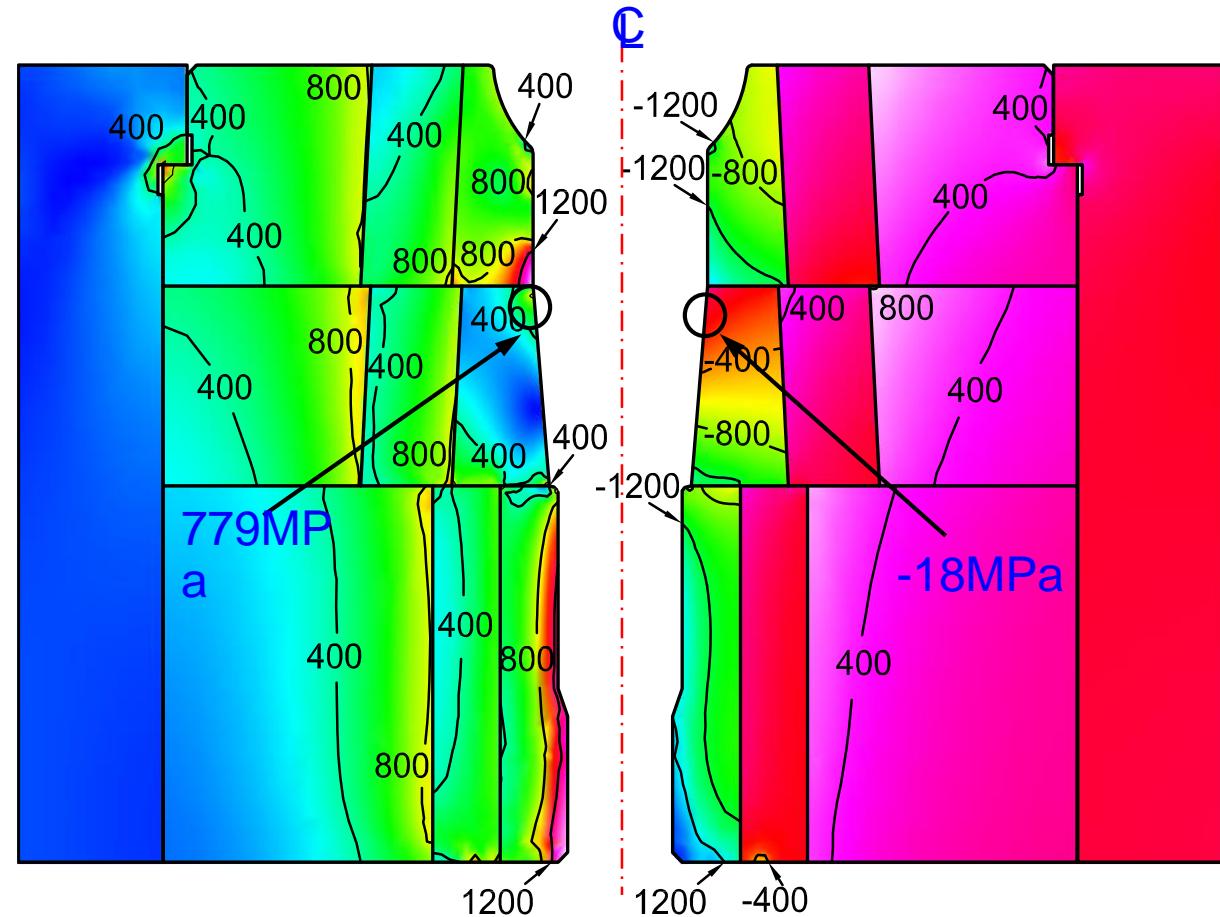
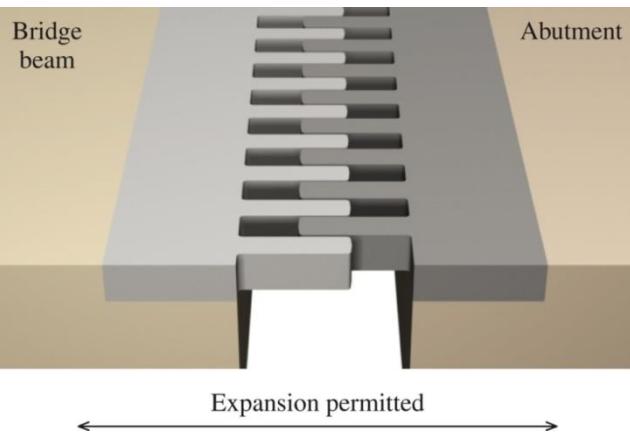
$\left. \begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha\Delta T \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] + \alpha\Delta T \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha\Delta T \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy}, \gamma_{yx} = \frac{1}{G} \tau_{yz}, \gamma_{zx} = \frac{1}{G} \tau_{zx} \end{aligned} \right\}$

Coefficient of thermal expansion

Hooke's law for an isotropic material



Thermal load and shrink fit





3.5 Solid Mechanics of Slender Members

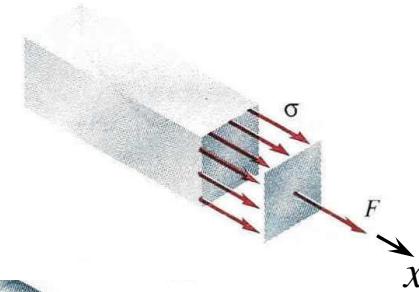


Mechanics of slender members

○ Truss member or rod – Uniaxial loading

- Axial force

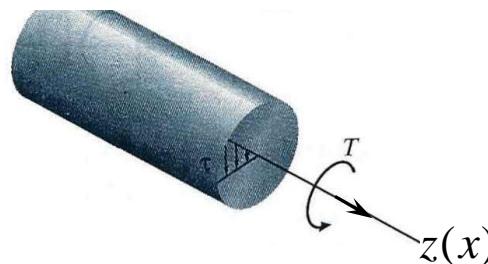
- Normal stress: $\sigma_x = \sigma_{xx} = \frac{F}{A}$



○ Circular shaft - Torsion

- Twisting moment

- Shear stress: $\tau_{z\theta} = \frac{Tr}{J}$

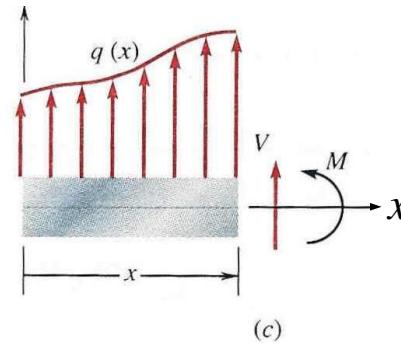
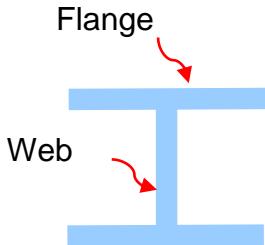


○ Beam

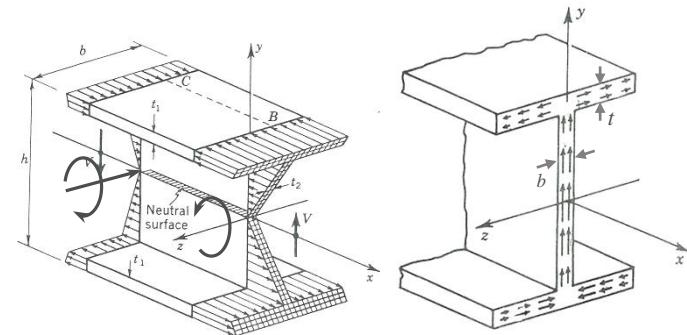
- Lateral force – Bending moment and shear force

- Normal and shear stress,

- I-beam, H-beam



● Stresses in beam

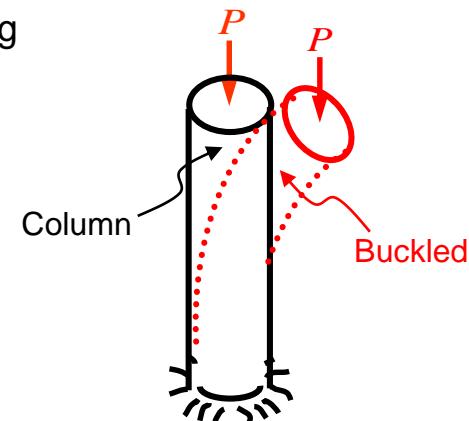


$$\sigma_x = -\frac{M_b y}{I_{zz}}, \quad \tau_{xy} = \frac{VQ}{bI_{zz}}, \quad \tau_{xz} = \frac{VQ}{tI_{zz}}$$

○ Column

- Compressive axial force

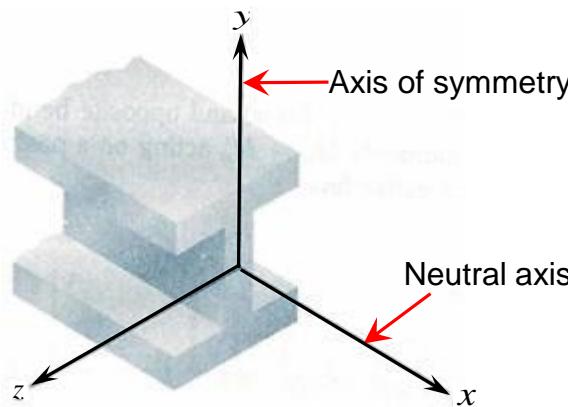
- Buckling



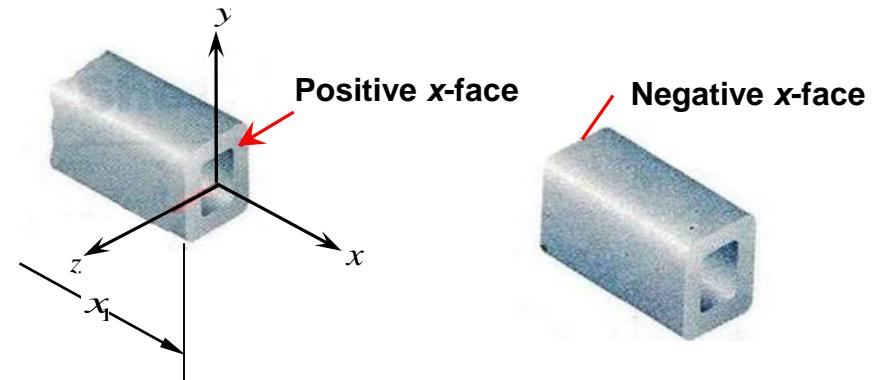


Internal force

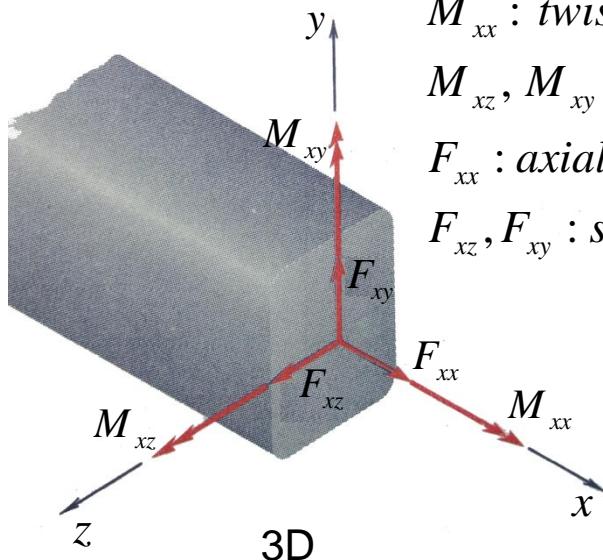
◎ Definition of axes



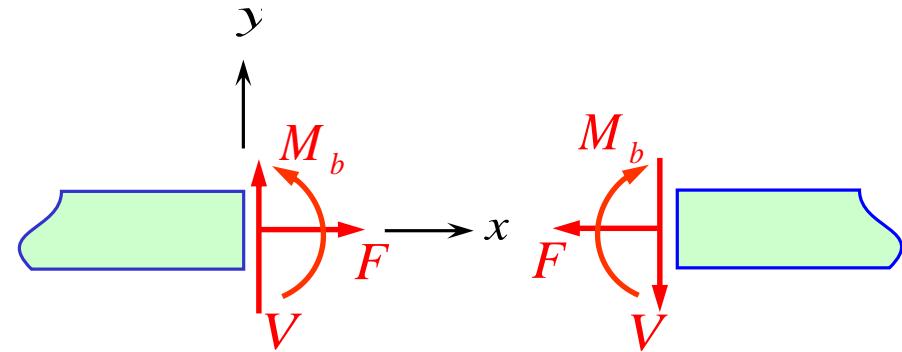
◎ Definition of cross-section



◎ Force and moment exerting cross-section



M_{xx} : twisting moment
 M_{xz}, M_{xy} : bending moment
 F_{xx} : axial force
 F_{xz}, F_{xy} : shear force



- ✓ First subscript: Direction of face
- ✓ Second subscript: Direction of force

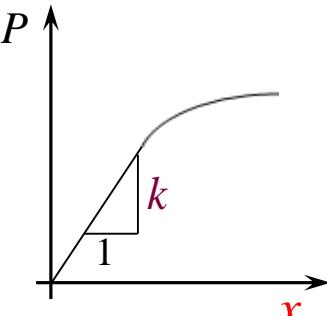
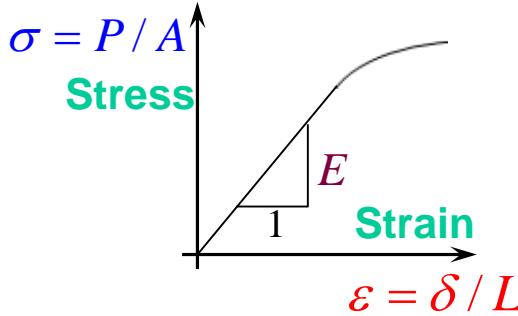
2D



3.6 Uniaxial Loading

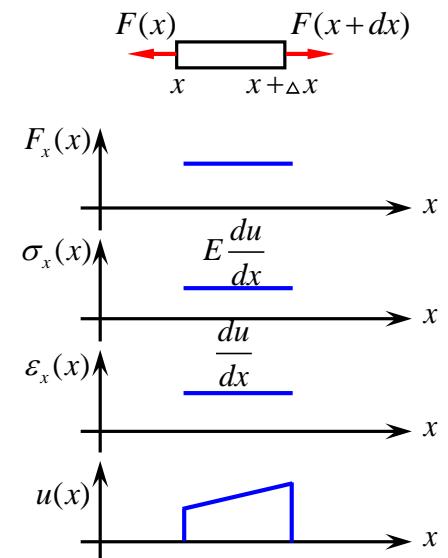
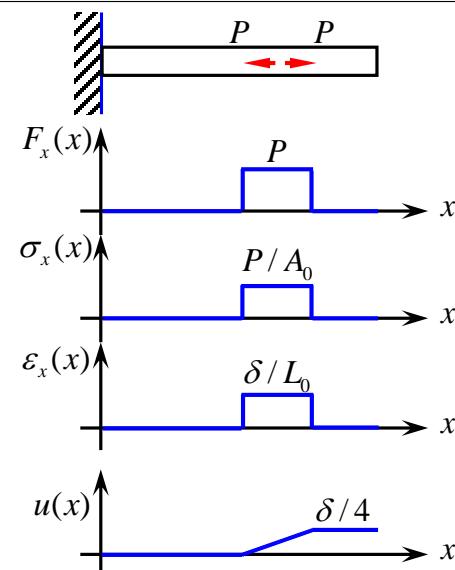
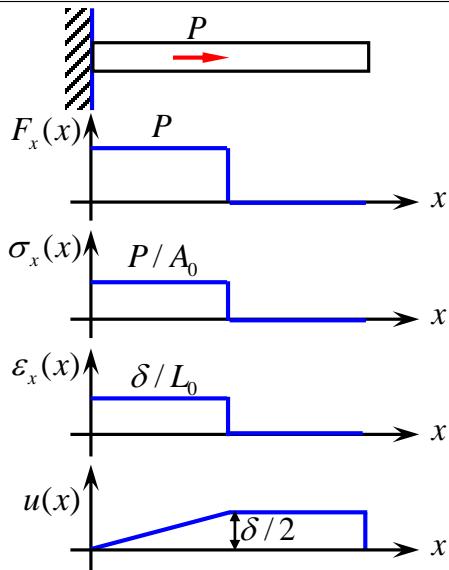
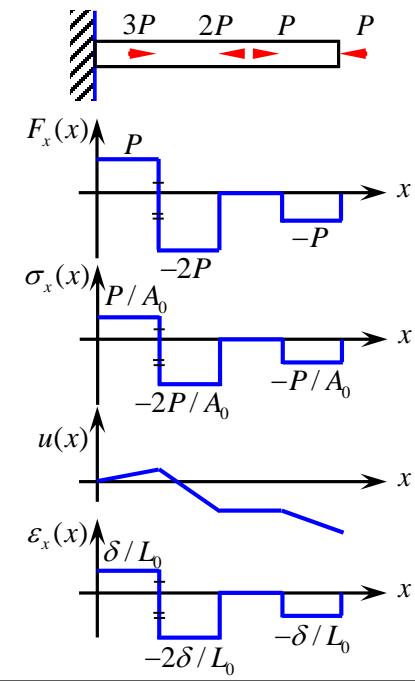
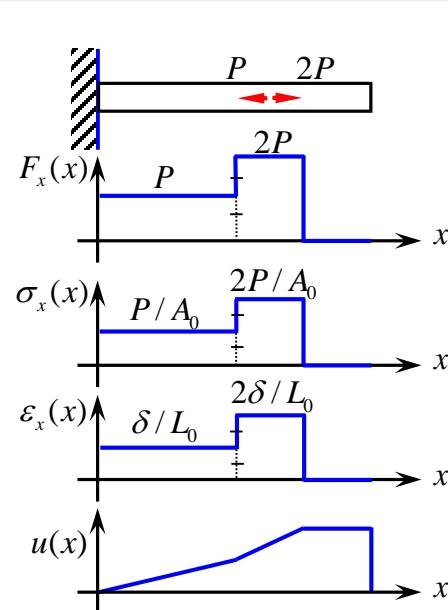
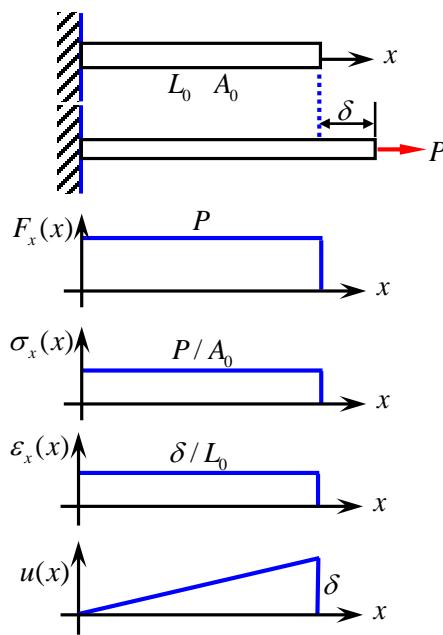


Deformation and stress in uniaxial loading

Spring)		Rod, truss member
$P \leftarrow -\text{spring} \rightarrow P, x$		$P \leftarrow \text{Rod} \rightarrow P, \delta$
 k : spring constant	Force-deformation relationship	$\sigma = P / A$ Stress  E : modulus of elasticity $\epsilon = \delta / L$ Strain
$P = kx$	Hooke's law $k_{eq} = \frac{AE}{L}$	$P = \frac{AE}{L} \delta, \sigma = E \epsilon, \frac{P}{A} = E \frac{\delta}{L}$
$x = \frac{1}{k} P$	Displacement	$\delta = \frac{L}{AE} P$
$U = \frac{1}{2} kx^2 = \frac{1}{2} \frac{1}{k} P^2$	Strain energy	$U = \frac{1}{2} k_{eq} \delta^2 = \frac{1}{2} \frac{L}{AE} P^2$

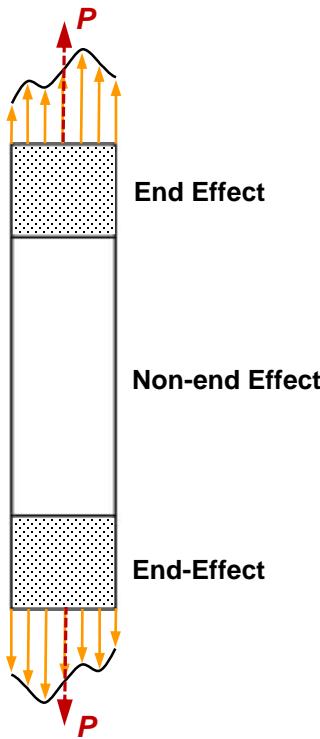


Displacement, strain and stress in rod



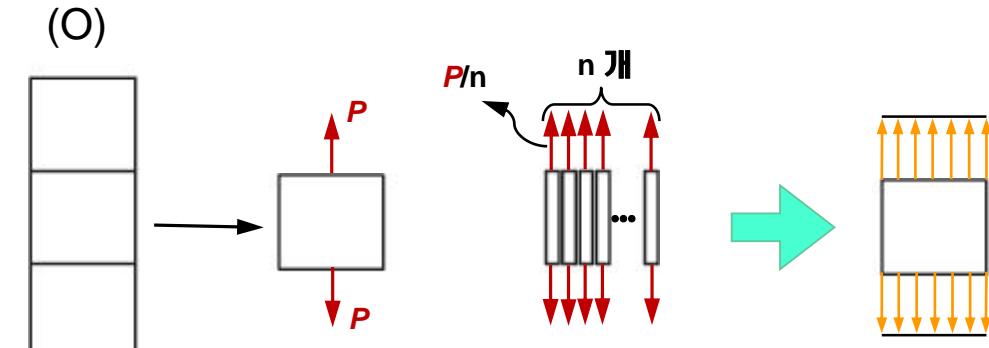
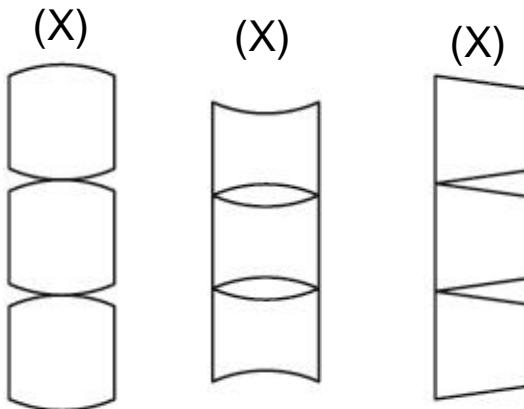
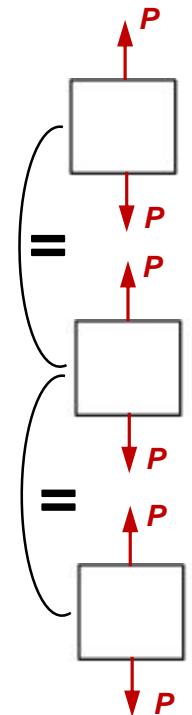
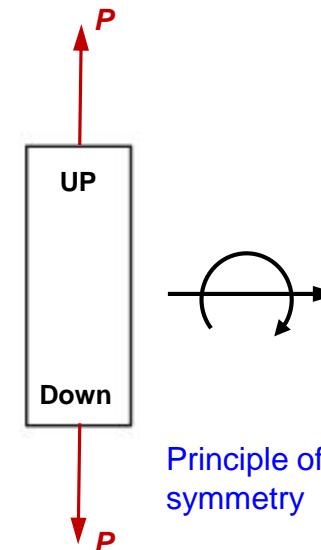


Rod



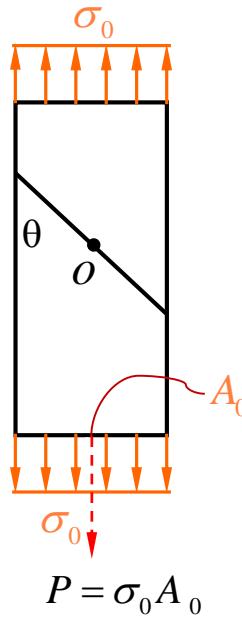
Saint-Venant's principle

- Two statically equivalent loads have nearly the same influence on the material except the region near to the load exerting area.





Internal force, stress vector in tensile test



$$\vec{P} = \sigma_0 A_0 \vec{i}$$

$n = \cos \theta \vec{i} + \sin \theta \vec{j}$

$A = \frac{A_0}{\sin \theta}$

x

y

$$P = \sigma_0 A_0$$

Stress vector

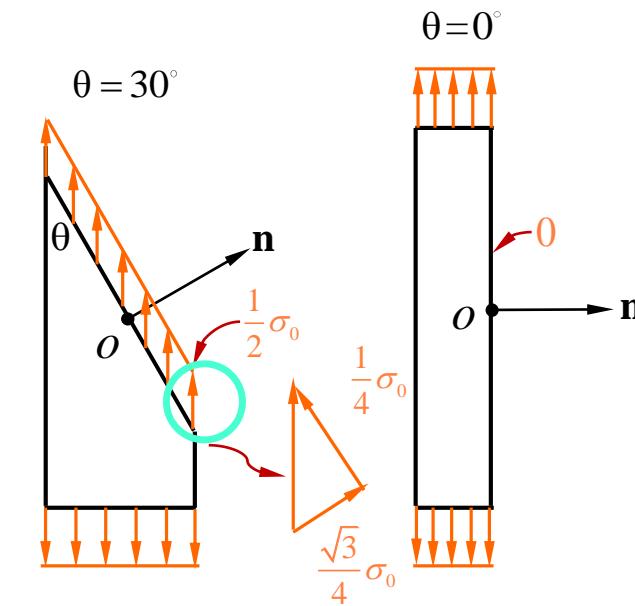
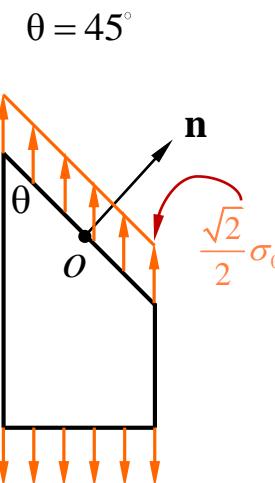
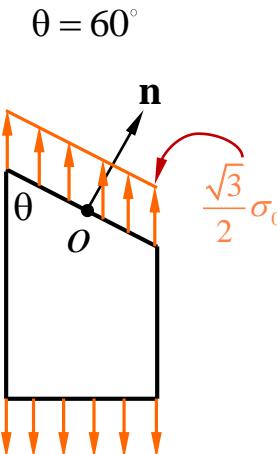
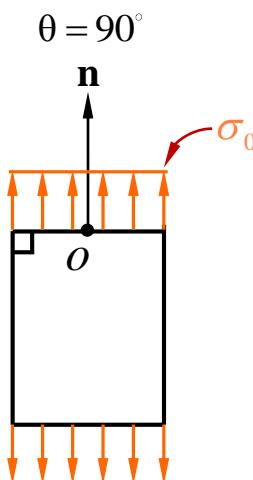
$$\overset{(n)}{\mathbf{T}} = \mathbf{t}^{(n)} = \frac{\vec{P}}{A} = \sigma_0 \sin \theta \vec{i}$$

$\sigma_0 \sin \theta$

$\sigma_0 \cos \theta \sin \theta \triangleq \sigma_{nt}$

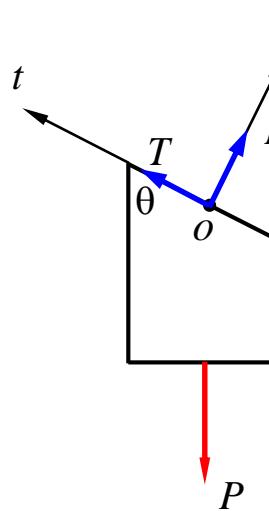
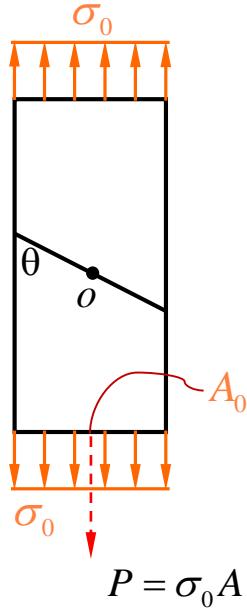
$\sigma_0 \sin^2 \theta \triangleq \sigma_{nn}$

$$\mathbf{t}^{(n)} = \sigma_0 \sin^2 \theta \vec{n} + \sigma_0 \cos \theta \sin \theta \vec{t}$$





Internal force and stress components



$$N = P \sin \theta$$

$$T = P \cos \theta$$

$$A = \frac{A_0}{\sin \theta}$$

Normal stress

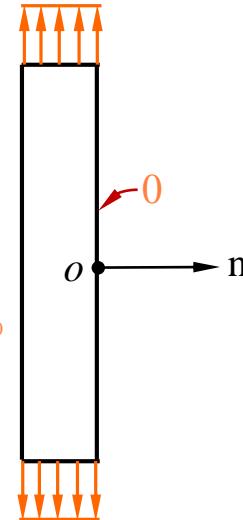
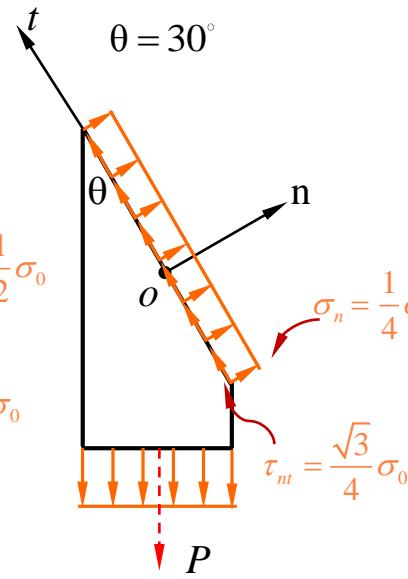
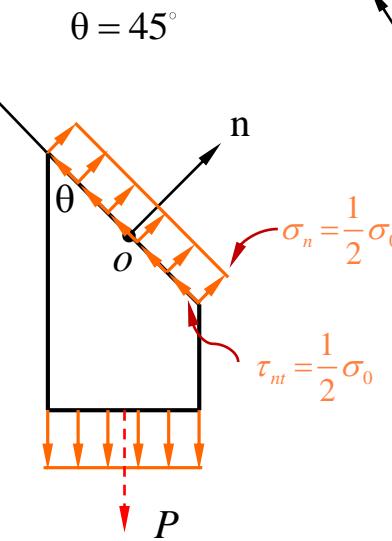
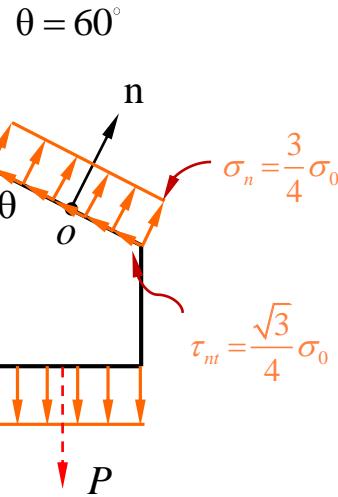
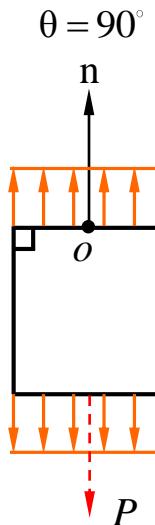
$$\sigma_{nn} = \frac{N}{A} = \frac{P}{A_0} \sin^2 \theta = \sigma_0 \sin^2 \theta$$

Shear stress

$$\tau_{nt} = \frac{T}{A} = \frac{P}{A_0} \cos \theta \sin \theta = \sigma_0 \cos \theta \sin \theta$$

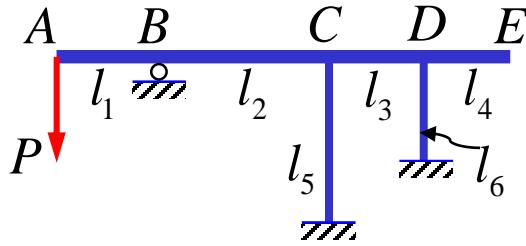
$$\mathbf{t}^{(n)} = \sigma_0 \sin^2 \theta \vec{n} + \sigma_0 \cos \theta \sin \theta \vec{t} = \sigma_{nn} \vec{n} + \sigma_{nt} \vec{t}$$

$$\theta = 0^\circ$$

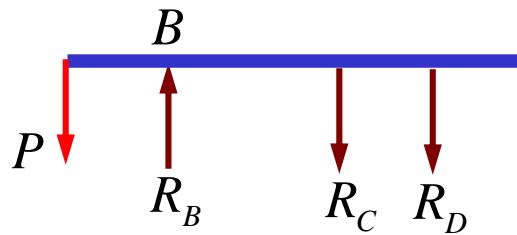




Example



<F. B. D.>



① Force equilibrium

$$\sum F_y = -P + R_B - R_C - R_D = 0 \quad ①$$

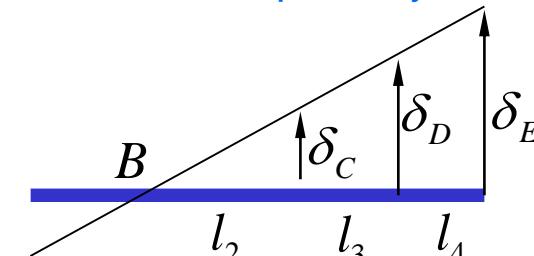
$$\sum M_B = l_1 P - l_2 R_C - (l_2 + l_3) R_D = 0 \quad ②$$

* x-directional displacement is neglected under small deformation.

③ Force-deformation relationship

$$\delta_C = \frac{R_C l_5}{A_C E_C}, \quad \delta_D = \frac{R_D l_6}{A_D E_D}$$

④ Geometric compatibility



$$l_2 : \delta_C = (l_2 + l_3) : \delta_D \rightarrow \delta_D = \left(1 + \frac{l_3}{l_2}\right) \delta_C \quad ⑤$$

○ Reaction and displacement of point E

$$\text{③④} \rightarrow \text{⑤:} \quad R_D = \frac{A_D R_C l_5}{A_C R_D l_6} \left(1 + \frac{l_3}{l_2}\right) R_C \quad ⑥$$

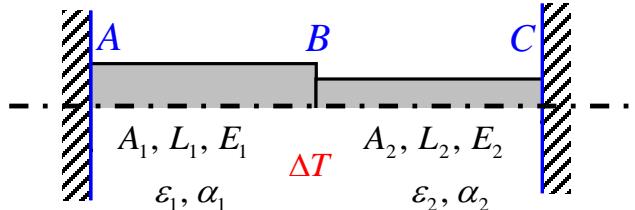
$$\text{①, ②, ⑥} \rightarrow \quad R_B, R_C, R_D$$

$$l_2 : \delta_C = (l_2 + l_3 + l_4) : \delta_E$$

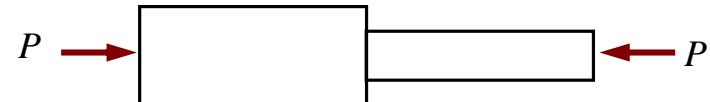


Statically indeterminate system – thermal load

< Method I >



<F. B. D.>



III Force-deformation relationship

$$\delta_{BA} = \varepsilon_1 L_1 = (-P/EA_1 + \alpha\Delta T)L_1$$

$$\delta_{CB} = \varepsilon_2 L_2 = (-P/EA_2 + \alpha\Delta T)L_2$$

$$\delta_{CA} = \delta_{BA} + \delta_{CB}$$

$$= -P\left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2}\right) + \alpha\Delta T(L_1 + L_2) \quad \textcircled{2}$$

II Geometric compatibility

$$\delta_{CA} = 0 \quad \textcircled{1}$$

○ From Eqs. $\textcircled{1}$ and $\textcircled{2}$

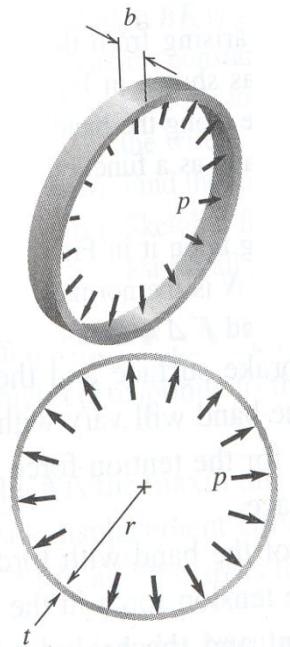
$$P = \frac{E\alpha\Delta T(L_1 + L_2)}{\left(\frac{L_1}{A_1} + \frac{L_2}{A_2}\right)} = \frac{A_1 A_2 E \alpha \Delta T (L_1 + L_2)}{L_1 A_2 + L_2 A_1}$$

< Method II >

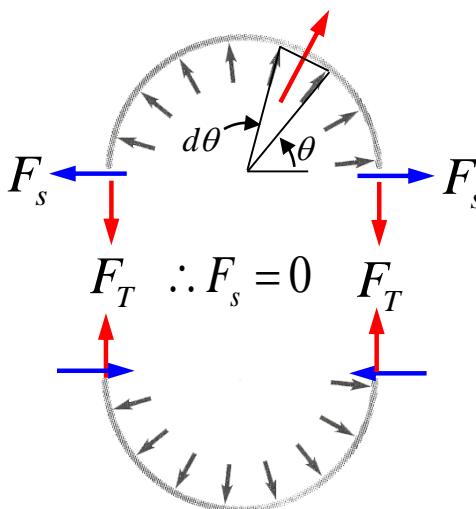
$$\begin{aligned} \delta_{CA} \text{ due to } \Delta T &= \alpha\Delta T(L_1 + L_2) \\ \delta_{CA} \text{ due to } P &= -\frac{PL_1}{AE} - \frac{PL_2}{A_2 E} \end{aligned} \quad \triangleright \delta_{CA}(\Delta T) = -\delta_{CA}(P)$$



Deformation of cylindrical pressure vessel



<F. B. D.>



① Force equilibrium

$$\sum F_x = 0 ;$$

< Method I >

$$-2F_T + \int_0^\pi rpb \sin \theta d\theta = 0$$

< Method II >

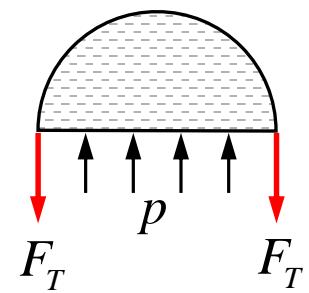
$$-2F_T + p \cdot b \cdot 2r = 0 \rightarrow F_T = pbr$$

② Force-deformation relationship

$$\delta_T = \frac{(pbr)2\pi(r + \frac{t}{2})}{(bt)E} \rightarrow$$

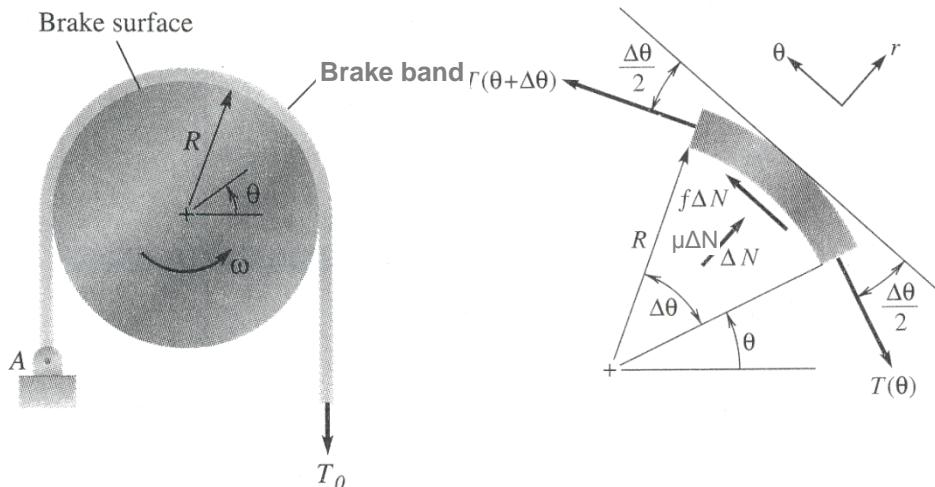
$$\delta_R = \frac{pr(r + \frac{t}{2})}{tE}$$

$$2\pi(r + \delta_R) = 2\pi r + 2\pi\delta_R \rightarrow \delta_R = \frac{\delta_T}{2\pi}$$





Tension in belt



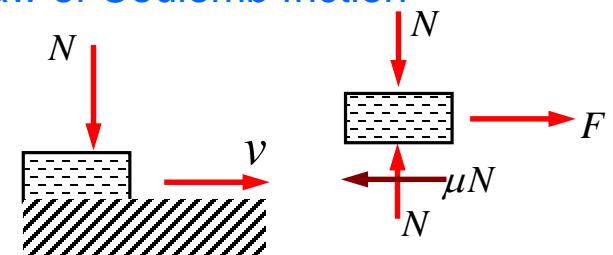
○ Applying RE on the FBD and governing equation

$$\begin{cases} \sum F_r = 0 ; \Delta N - T \sin(\Delta\theta/2) - T(\theta + \Delta\theta) \sin(\Delta\theta/2) = 0 \\ \sum F_\theta = 0 ; \mu \Delta N - T \cos(\Delta\theta/2) + T(\theta + \Delta\theta) \cos(\Delta\theta/2) = 0 \end{cases}$$

$$\downarrow T(\theta + \Delta\theta) = T(\theta) + \frac{dT}{d\theta} \Delta\theta, \sin \frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}, \cos \frac{\Delta\theta}{2} \approx 1$$

$$\begin{pmatrix} \Delta N - T \Delta\theta - T' \frac{\Delta\theta^2}{2} = 0 \\ \mu \Delta N + T' \Delta\theta = 0 \end{pmatrix} \xrightarrow{\Delta\theta \rightarrow 0} \begin{pmatrix} \frac{dN}{d\theta} = T \\ \frac{dN}{d\theta} = -\frac{T'}{\mu} \end{pmatrix} \xrightarrow{\underline{\underline{\frac{dT}{d\theta} = -\mu T}}}$$

○ Law of Coulomb friction



- Magnitude: $F = \mu N$
- Direction: Preventing relative motion
- μ : Coefficient of Coulomb friction,

○ Solving

$$\frac{dT}{T} = -\mu d\theta$$

$$\ln T = C - \mu \theta$$

$$\underline{\underline{T = C e^{-\mu \theta}}}$$

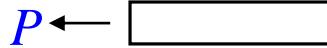
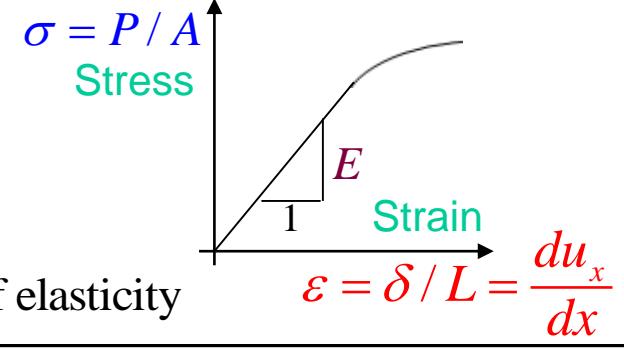
○ Boundary condition

$$T(0) = T_0$$

$$\rightarrow T = T_0 e^{-\mu \theta}$$



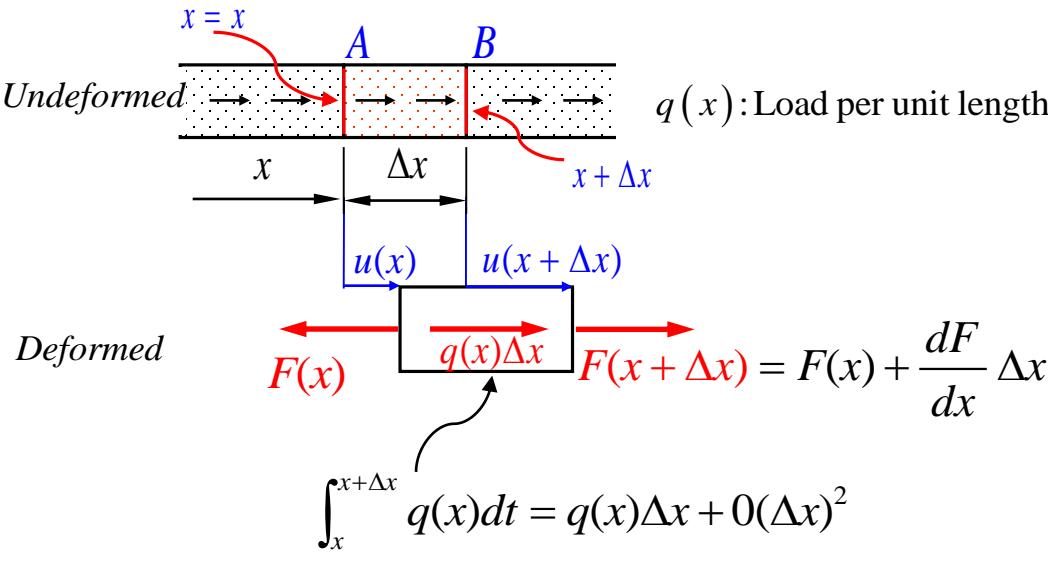
Strain energy in rod

Force-deformation relationship	<p>$P \leftarrow$  $\rightarrow P, \delta$</p> <p>$P \leftarrow$  $\rightarrow P, x$</p> <p>k_{eq}</p> <p>E: modulus of elasticity</p>  <p>$\sigma = P / A$ Stress</p> <p>$\varepsilon = \delta / L = \frac{du_x}{dx}$ Strain</p>
Hooke's law	$\sigma = E\varepsilon, \frac{P}{A} = E \frac{\delta}{L}, P = \frac{AE}{L} \delta, \delta = \frac{PL}{AE}, k_{eq} = \frac{AE}{L}$
Strain energy	$U = \frac{1}{2} k_{eq} \delta^2 = \frac{P^2 L}{2AE} = \frac{1}{2} \frac{P}{A} \left(\frac{PL}{AE} / L \right) AL = \frac{1}{2} \frac{P}{A} \frac{\delta}{L} V$ <p>$u = \frac{U}{V} = \frac{1}{2} \sigma \varepsilon \leftarrow \text{strain energy density (function)}$</p> $U = \int_V u dV = \frac{1}{2} \int_V \sigma \varepsilon dV = \frac{1}{2} \int_V \frac{\sigma^2}{E} dV$ $= \frac{1}{2} \int_L \frac{P^2}{A^2 E} \int_A dA dx = \frac{1}{2} \int_L \frac{P^2}{AE} dx$ $U = \frac{1}{2} \int_V E \varepsilon^2 dV = \frac{1}{2} \int_L E \frac{du_x}{dx}^2 \int_A dA dx = \frac{1}{2} \int_L EA \frac{du_x}{dx}^2 dx$



Generalization of uniaxial loading

○ Strain in rod under uniaxial loading



$$\varepsilon = \frac{A'B' - AB}{AB} = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} = \frac{du}{dx} = \varepsilon_x$$

$$\begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & -v\varepsilon_x & 0 \\ 0 & 0 & -v\varepsilon_x \end{pmatrix}$$

I Force equilibrium

$$\sum F_x = -F(x) + q(x)\Delta x + F(x + \Delta x) = 0$$

$$\rightarrow \frac{dF}{dx} = -q(x)$$

III Stress-strain relation

$$\sigma = E\varepsilon \quad \text{Hooke's law}$$

$$F(x) = A\sigma = AE\varepsilon = AE \frac{du}{dx}$$

$$\frac{dF}{dx} = -q(x)$$

$$\therefore \frac{d}{dx} \left(AE \frac{du}{dx} \right) = -q(x)$$

↑ Governing equation



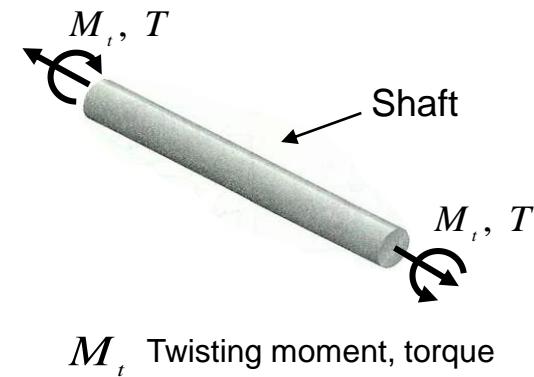
3.7 Torsion



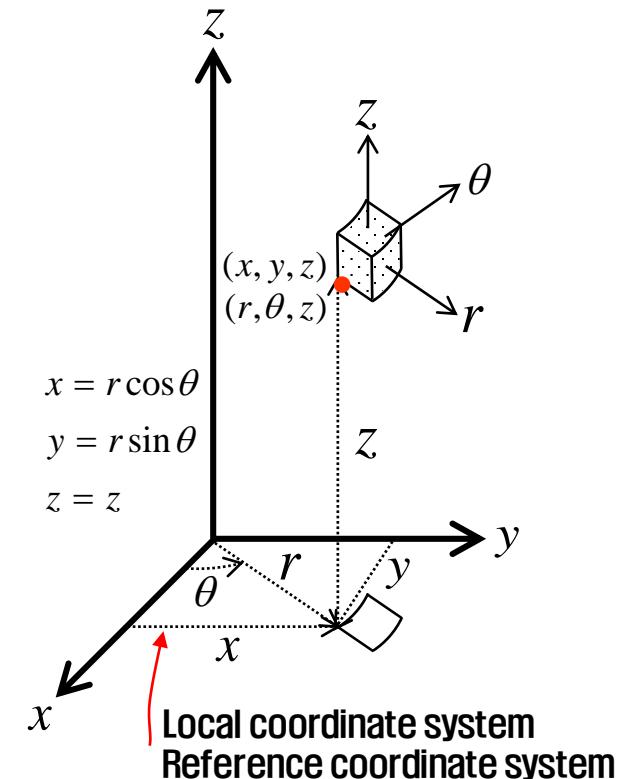
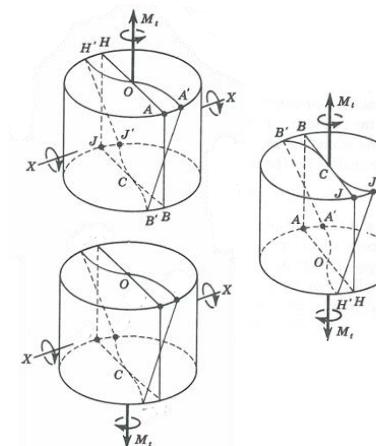
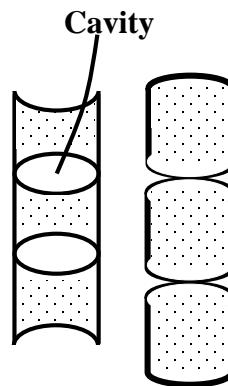
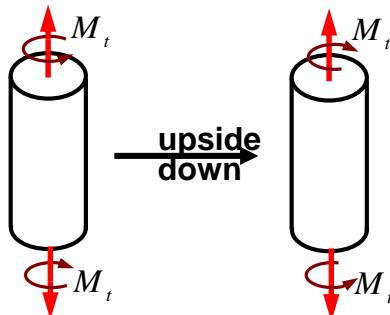
Torsion of cylindrical shaft, problem definition

- Cross-section: Circular shaft (CS)
- Role of CS: Power transmission, spring, etc.
- Assumption to apply rule of symmetry

- End-effects are negligible (Saint-Venant Principle)
- Uniform cross-section
- Geometry and material are axisymmetric
- Symmetric expansion and contraction are neglected
- Lengthening and shortening are neglected



- Rule of symmetry



- Summary

- Diametral straight line remains straight line
- Plane section, perpendicular to the central line, remains plane

Relation of strain and angle of twist

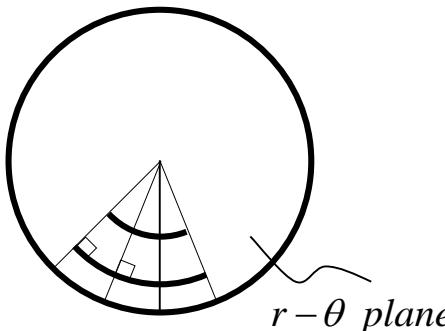
○ Strain components

$$\begin{pmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{pmatrix} \rightarrow \begin{pmatrix} \epsilon_{rr} & \gamma_{r\theta} & \gamma_{rz} \\ \gamma_{\theta r} & \epsilon_{\theta\theta} & \gamma_{\theta z} \\ \gamma_{zr} & \gamma_{z\theta} & \epsilon_{zz} \end{pmatrix}$$

○ Normal strain: $\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{zz} = 0$

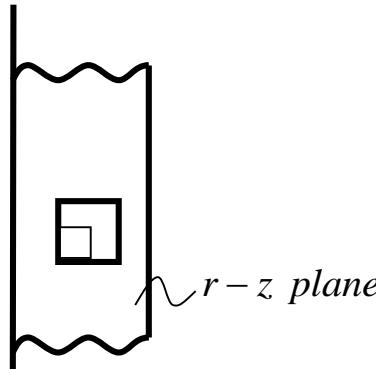
From assumption

○ Shear strain: $\gamma_{r\theta} = 0$



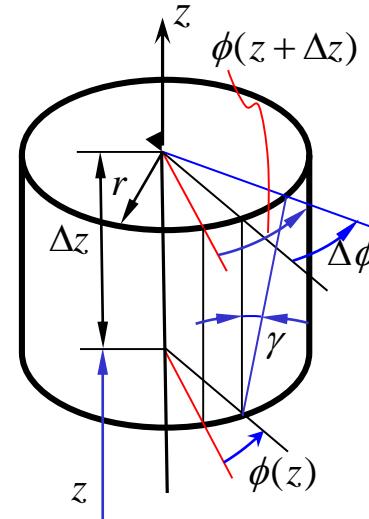
$\gamma_{r\theta} = 0$

○ Shear strain: $\gamma_{rz} = 0$



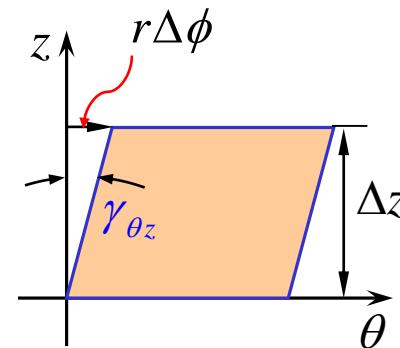
$\gamma_{rz} = 0$

○ Shear strain $\gamma_{\theta z}$ and angle of twist



ϕ : Angle of twist

$\frac{d\phi}{dz}$: Rate of twist



$$r \Delta \phi = \gamma_{\theta z} \Delta z \rightarrow \gamma_{\theta z} = r \frac{\Delta \phi}{\Delta z}$$

$\gamma_{\theta z} = \gamma_{z\theta} = r \frac{d\phi}{dz}$

- ϕ : Angle of twist

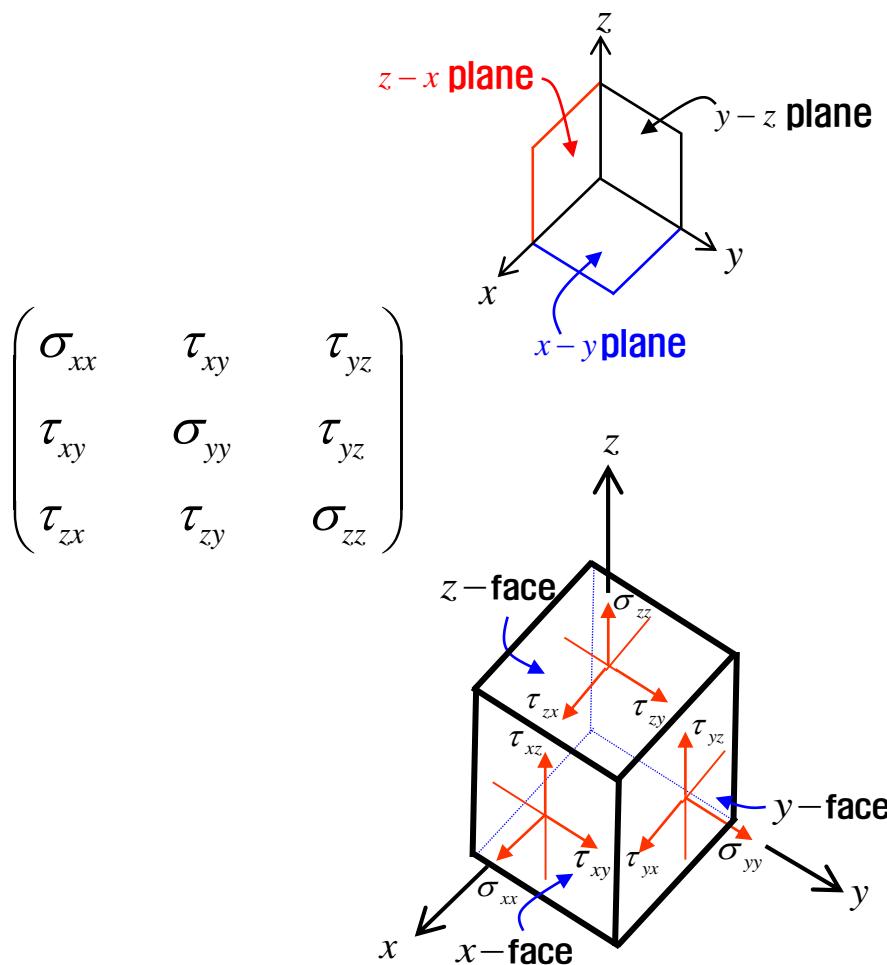
- $\frac{d\phi}{dz}$: Rate of twist

$\gamma_{rx} = 0$

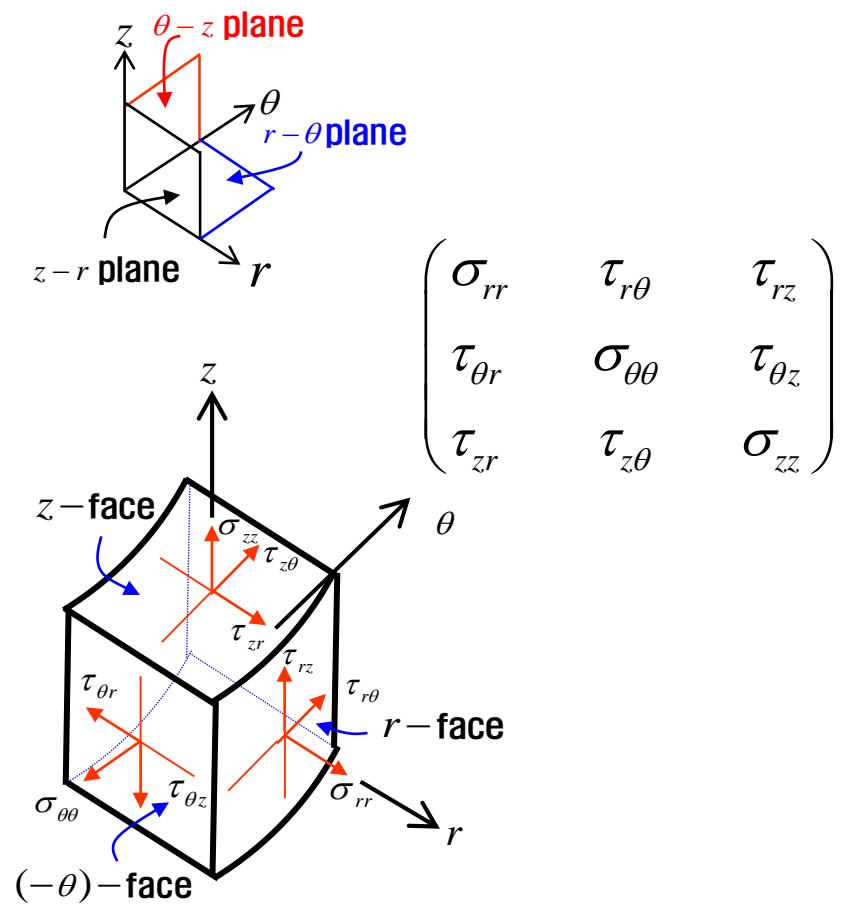


Stress components

◎ x-y-z coordinate system



◎ Cylindrical coordinate system





Hooke's law

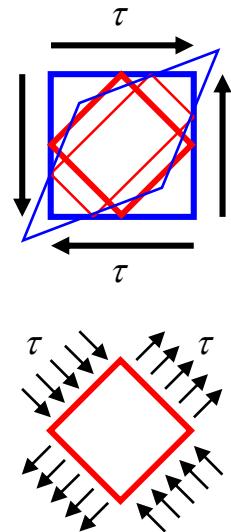
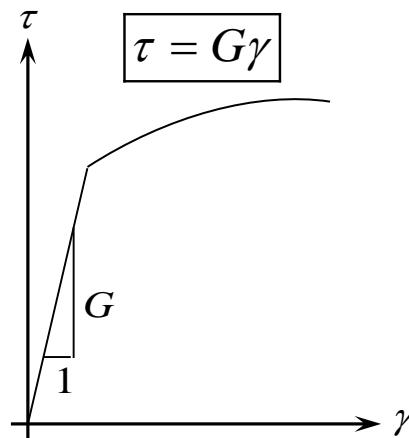
$$\tau_{\theta z} = G \gamma_{\theta z} = Gr \frac{d\phi}{dz} = Gr \frac{\phi_L}{L}$$

$$\sigma_{xx} = E \varepsilon_{xx} = E \frac{du}{dx} = E \frac{u_L}{L}$$

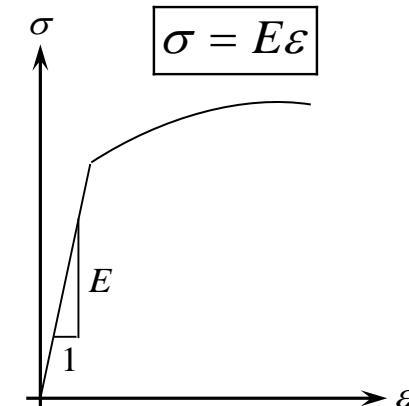
$$\cdot \varepsilon_{rr} = \frac{1}{E} [\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})]$$

$$G = \frac{E}{2(1+\nu)}$$

○ Torsion test



○ Tensile test



$$\cdot \varepsilon_{\theta\theta} = \frac{1}{E} [\sigma_{\theta\theta} - \nu(\sigma_{zz} + \sigma_{rr})]$$

$$\cdot \varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})]$$

$$\cdot \gamma_{r\theta} = \frac{2(1+\nu)}{E} \sigma_{r\theta} = \frac{1}{G} \sigma_{r\theta}$$

$$\cdot \gamma_{\theta z} = \frac{2(1+\nu)}{E} \sigma_{\theta z} = \frac{1}{G} \sigma_{\theta z}$$

$$\cdot \gamma_{zr} = \frac{2(1+\nu)}{E} \sigma_{zr} = \frac{1}{G} \sigma_{zr}$$

$$\begin{pmatrix} \varepsilon_{rr} & \gamma_{r\theta} & \gamma_{rz} \\ \gamma_{\theta r} & \varepsilon_{\theta\theta} & \gamma_{\theta z} \\ \gamma_{zr} & \gamma_{z\theta} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & rd\phi/dz \\ 0 & rd\phi/dz & 0 \end{pmatrix} \xrightarrow{\hspace{1cm}} \begin{pmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Grd\phi/dz \\ 0 & Grd\phi/dz & 0 \end{pmatrix}$$



Requirement on equilibrium

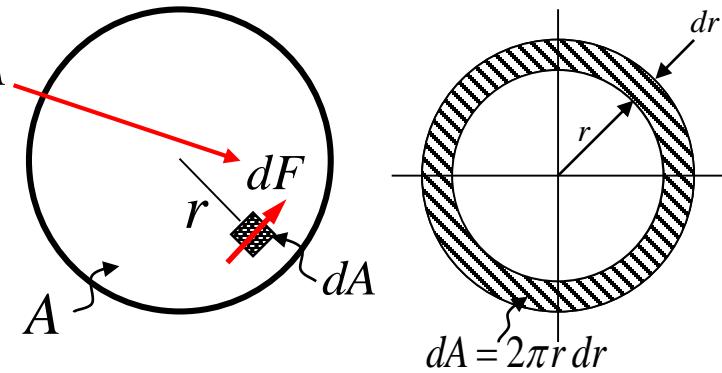
◎ Homogeneous shaft

$$M_t = \int_A r dF = \int_A r \tau dA = \frac{d\phi}{dz} \int_A Gr^2 dA \quad \leftarrow \quad dF = \tau dA$$

$$M_t = \frac{d\phi}{dz} G \int_A r^2 dA \triangleq \frac{d\phi}{dz} GJ \quad \leftarrow \quad GJ : \text{Torsional rigidity}$$

$$\frac{d\phi}{dz} = \frac{M_t}{GJ} \rightarrow \tau_{\theta z} = \tau_{z\theta} = \frac{M_t r}{J}$$

$$J = \int_A r^2 dA = 2\pi \int_{R_i}^{R_0} r^3 dr = \frac{\pi}{2} (R_0^4 - R_i^4) \quad \leftarrow \quad \text{Polar second moment of inertia at the cross-section}$$

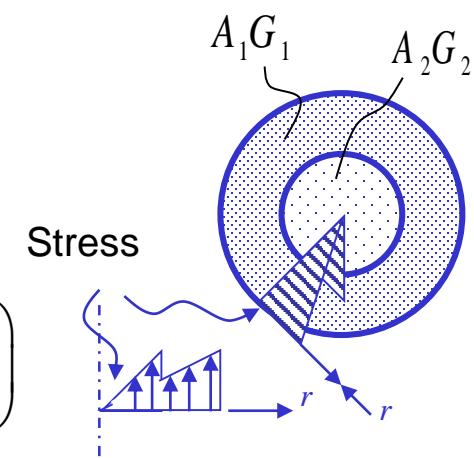


◎ Composite shaft

$$M_t = \int_A r dF = \int_A r \tau dA = \frac{d\phi}{dz} \int_A Gr^2 dA$$

$$M_t = \frac{d\phi}{dz} \left[G_1 \int_{A_1} r^2 dA + G_2 \int_{A_2} r^2 dA \right] = \frac{d\phi}{dz} (G_1 J_1 + G_2 J_2)$$

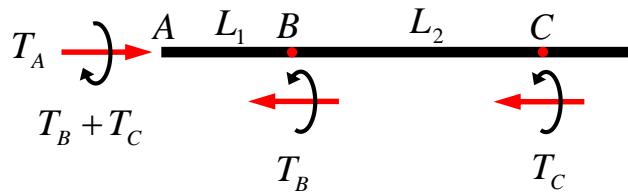
$$\frac{d\phi}{dz} = \frac{M_t}{G_1 J_1 + G_2 J_2} \rightarrow \tau = \frac{G_i r M_t}{G_1 J_1 + G_2 J_2}, \quad G_i = \begin{cases} G_1 & \text{if } r \in A_1 \\ G_2 & \text{if } r \in A_2 \end{cases}$$





Example 1 – Statically determinate system

< F.B.D. >



○ Moment balance

$$\sum M_x = 0;$$

$$T_A - T_B - T_C = 0 \rightarrow T_A = T_B + T_C$$

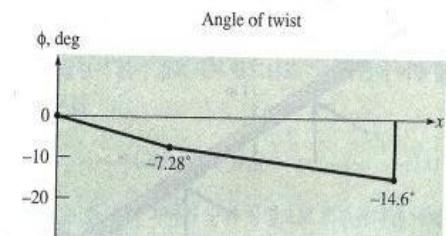
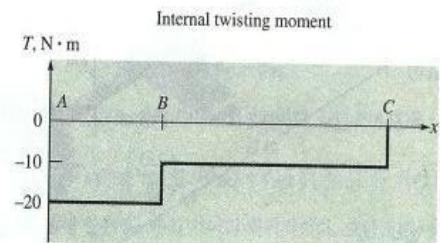
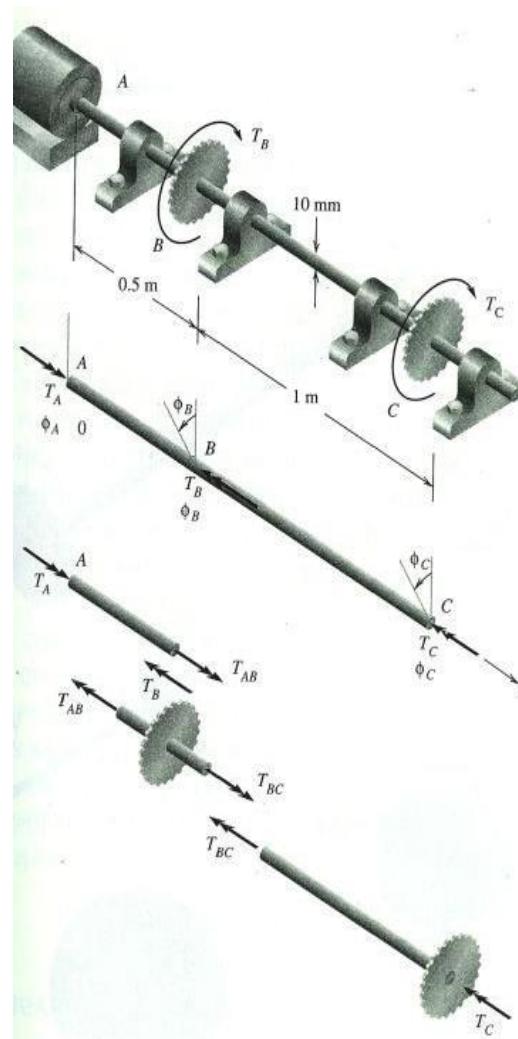
$$M_{t,AB} = -T_B - T_C$$

$$M_{t,BC} = -T_C$$

○ Maximum angle of twist and stress

$$\phi_{CA} = \frac{-1}{GJ} [(T_B + T_C)L_1 + T_C L_2]$$

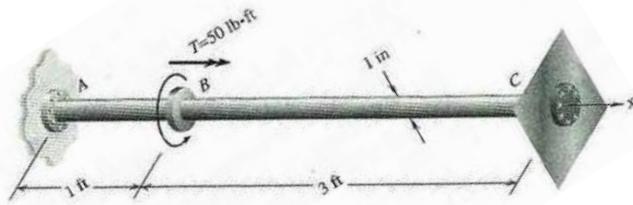
$$\tau_{\max} = \frac{(T_B + T_C)(d/2)}{J}$$



(d)



Example 2 – Statically indeterminate system



- Moment balance

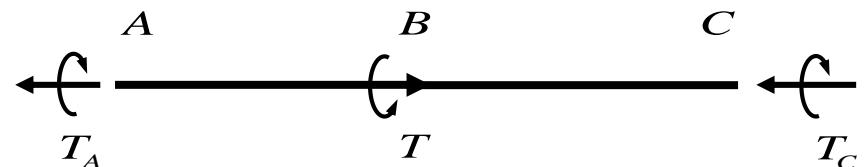
$$\sum M_x = 0;$$

$$-T_A + T - T_C = 0 \rightarrow T_C = T - T_A$$

$$M_{t,AB} = T_A$$

$$M_{t,BC} = -T_C = T_A - T$$

< F.B.D. >



- Geometric compatibility: $\phi_{CA} = 0$

$$\phi_{CA} = \phi_{BA} + \phi_{CB} = \frac{1}{GJ} [T_A L_1 + (T_A - T) L_2] = 0$$

$$\rightarrow T_A = \frac{L_2 T}{L_1 + L_2}$$

- Angle of twist at Point B and maximum shear stress

$$\phi_{BA} = \int \frac{M_{t,AB}}{GJ} dx = \frac{T_A L_1}{GJ} = \frac{1}{GJ} \left(\frac{L_1 L_2 T}{L_1 + L_2} \right) = \frac{L_1 L_2 T}{GJ(L_1 + L_2)}$$

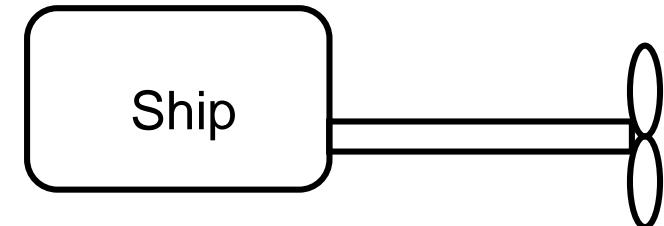
$$\tau_{\max} = \frac{T_{\max} r}{J}, \quad T_{\max} = \max(T_A, T_C)$$



Example 3 – Design of circular shaft

- Given value

$$P = 260 \text{ hp}, \quad n = 3800 \text{ rpm}, \quad \tau_a = 30,000 \text{ psi}$$



- Determination of diameter of the shaft

$$P = 260 \text{ hp} = 260 \times 6600 \text{ in} \cdot \text{lb/s} = 1.716 \times 10^6 \text{ in} \cdot \text{lb/sec}$$

$$\omega = 3800 \text{ rpm} = 3800 \times \frac{2\pi}{60} \frac{\text{rad}}{\text{sec}} = 398 \frac{\text{rad}}{\text{sec}}$$

$$P = T\omega \rightarrow T = 4.31 \times 10^3 \text{ in} \cdot \text{lb} \quad \text{--- ①}$$

$$\tau_{\max} = \frac{T(d/2)}{\frac{\pi}{32}d^4} = \frac{16T}{\pi d^3} = \tau_a (= 30 \times 10^3 \frac{\text{lb}}{\text{in}^2}) \rightarrow d = \sqrt[3]{\frac{16T}{\pi \tau_a}} \quad \text{--- ②}$$

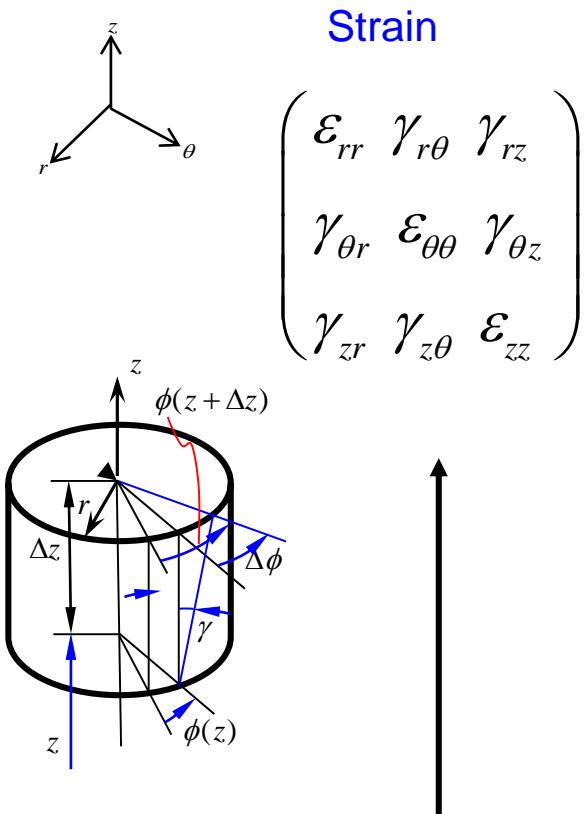
$$\textcircled{1} + \textcircled{2} \rightarrow d = \sqrt[3]{\frac{16T}{\pi \tau_a}} = 0.90 \text{ in}$$

$$\bullet 1 \text{hp} = 76 \text{kg} \cdot \text{m/s}$$

$$= 76 \times \frac{1}{0.453} \times \frac{1}{0.3048} \text{ lb} \cdot \text{ft/sec}$$



Summary of torsional problem of circular shaft



Hooke's law \rightarrow

$$\begin{aligned} \cdot \epsilon_{rr} &= \frac{1}{E} [\sigma_{rr} - \nu (\sigma_{\theta\theta} + \sigma_{zz})] \\ \cdot \epsilon_{\theta\theta} &= \frac{1}{E} [\sigma_{\theta\theta} - \nu (\sigma_{zz} + \sigma_{rr})] \\ \cdot \epsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{rr} + \sigma_{\theta\theta})] \\ \cdot \gamma_{r\theta} &= \frac{\tau_{r\theta}}{G} \\ \cdot \gamma_{\theta z} &= \frac{\tau_{\theta z}}{G} \\ \cdot \gamma_{zr} &= \frac{\tau_{zr}}{G} \end{aligned}$$

Force equilibrium $\rightarrow \frac{d\phi}{dz} = \frac{M_t}{GJ}$

Stress

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Gr \frac{d\phi}{dz} \\ 0 & Gr \frac{d\phi}{dz} & 0 \end{pmatrix}$$

$\tau_{z\theta} = \tau_{\theta z} = \frac{M_t r}{J}$

For composite shaft $\frac{d\phi}{dz} = \frac{M_t}{G_1 J_1 + G_2 J_2} + \dots$

- Rule of symmetry
 - Assumption: $\epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{zz} = 0$
 - Geometric compatibility:
- $$\gamma_{z\theta} = \gamma_{\theta z} = r \frac{d\phi}{dz}$$
- $$\gamma_{r\theta} = \gamma_{\theta r} = 0$$
- $$\gamma_{rz} = \gamma_{zr} = 0$$

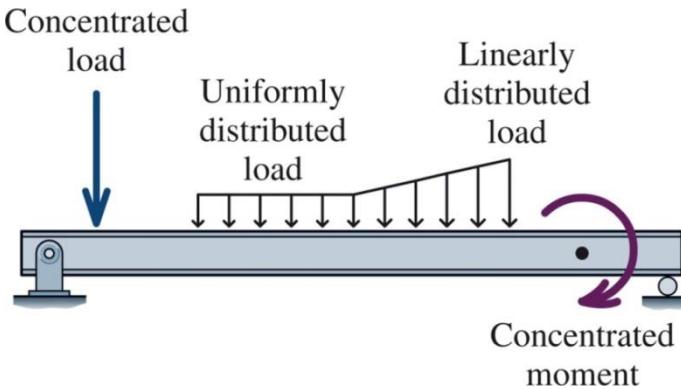
$$\phi = \int_L \frac{d\phi}{dz} dz = \int_L \frac{M_t}{GJ} dz$$

$$U = \frac{1}{2} \int_L \frac{M_t^2}{GJ} dz$$

3.8 Beam Theory



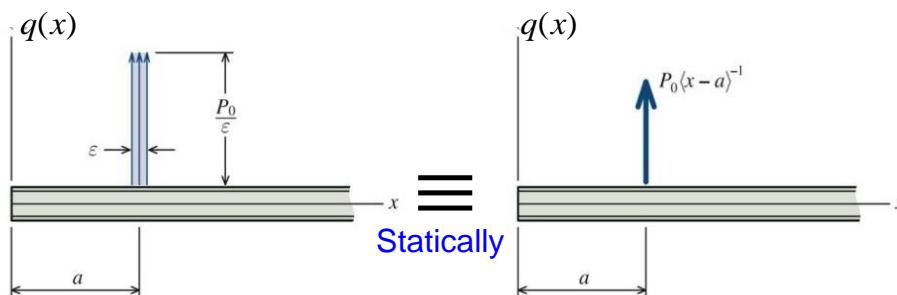
Beam and external load



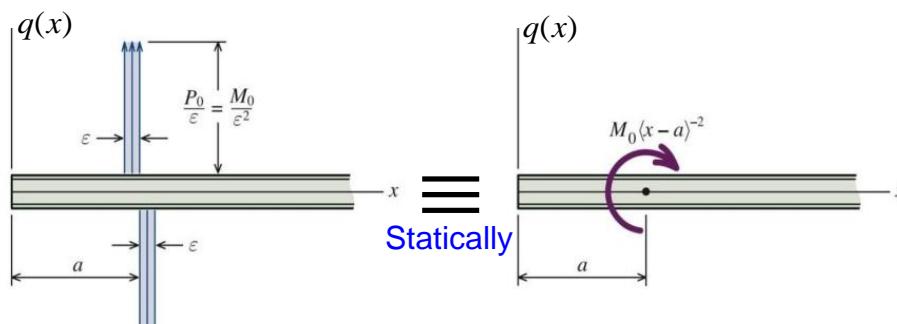
Resultant force

Statically the same concentrated force with a set of distributed force = Vector sum passing through centroid of loading diagram

○ Concentrated force



○ Concentrated moment



$q(x)$ = Load intensity function = Load/length

The top part shows a "Loading diagram" $q(x)$ plotted against position x , with a centroid at $X = \bar{x}$. The area under the curve is labeled R .

The bottom part shows a rectangular approximation of the loading diagram, with its centroid at \bar{x} .

$$R \triangleq \int_L q(x) dx$$

$$X R = \int_L x q(x) dx$$

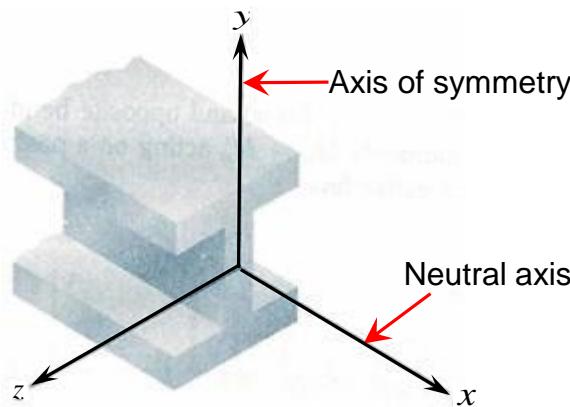
$$X = \frac{\int_L x q(x) dx}{\int_L q(x) dx} = \frac{\int_A x dA}{\int_A dA} = \bar{x}$$

$q(x)dx = dA$

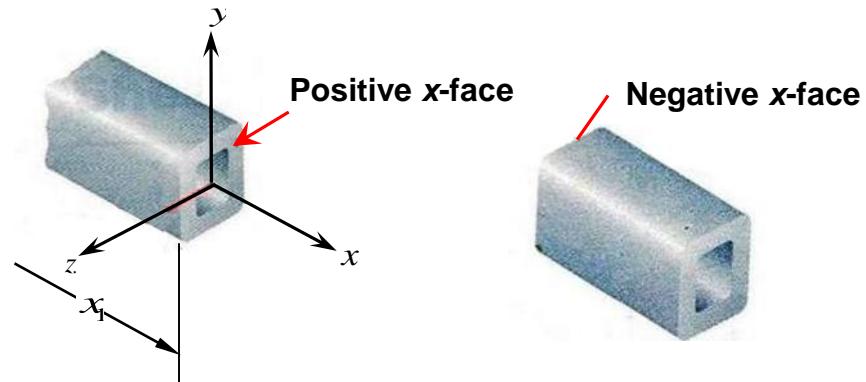


Internal force (Shear force and bending moment)

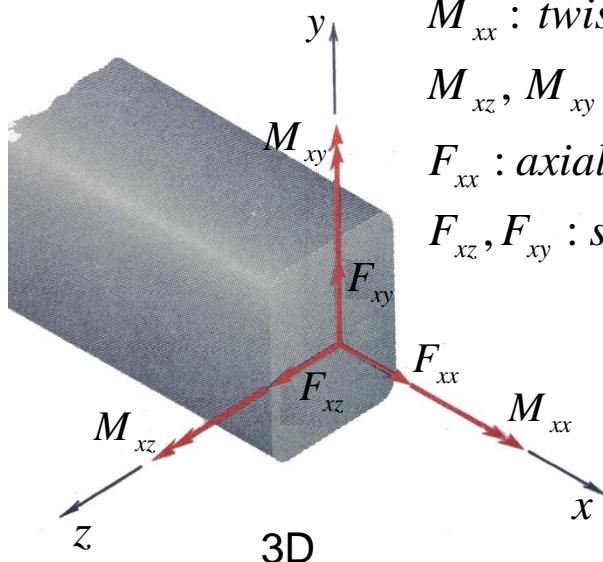
◎ Definition of axes



◎ Definition of cross-section



◎ Force and moment exerting cross-section

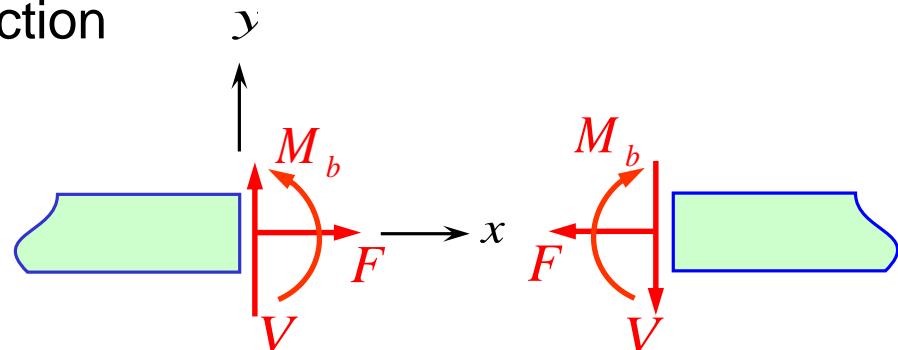


M_{xx} : twisting moment

M_{xz}, M_{xy} : bending moment

F_{xx} : axial force

F_{xz}, F_{xy} : shear force



$$F = F_{xx}, \quad V = F_{xy}, \quad M_b = M_{xz}$$

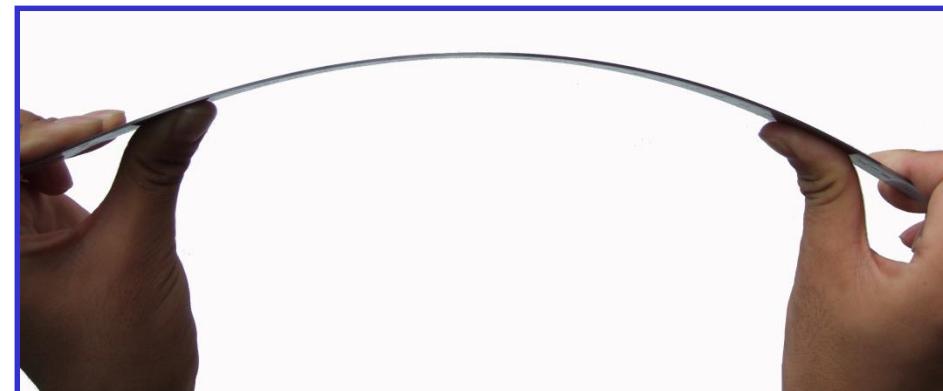
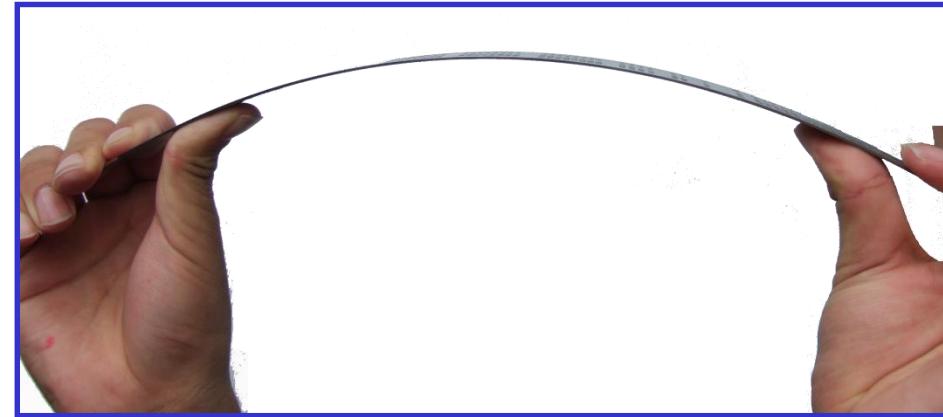
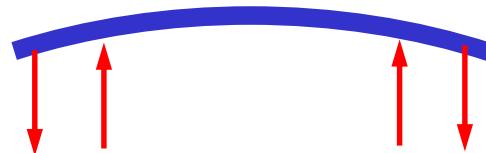
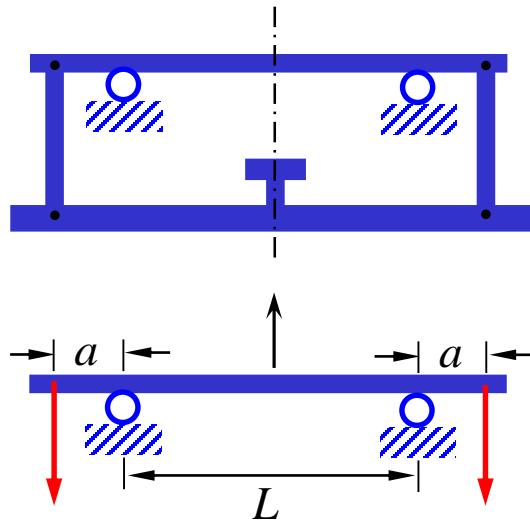
- ✓ First subscript: Direction of face
- ✓ Second subscript: Direction of force

2D



Pure bending

⊕ $V(x) = 0, \quad M_b(x) = \text{constant}$

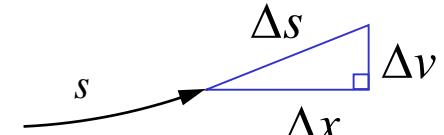
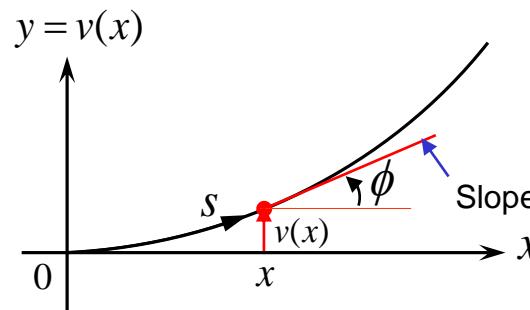




Derivation of relationship between κ and $v(x)$

- $\kappa = \frac{1}{\rho} \equiv \frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot \frac{dx}{ds}$

ϕ' ? **Curvature**



$$v'(x) = \tan \phi \rightarrow \cos^2 \phi = \frac{1}{1 + v'(x)^2}$$

$$v''(x) = \phi' \sec^2 \phi$$

$$\phi' = \frac{d\phi}{dx} = \frac{v''(x)}{1 + v'(x)^2}$$

- $\kappa = \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot \frac{dx}{ds} = \frac{v''(x)}{\left(1 + v'^2(x)\right)^{\frac{3}{2}}} \approx v''(x)$

$\circ \frac{dx}{ds}$?

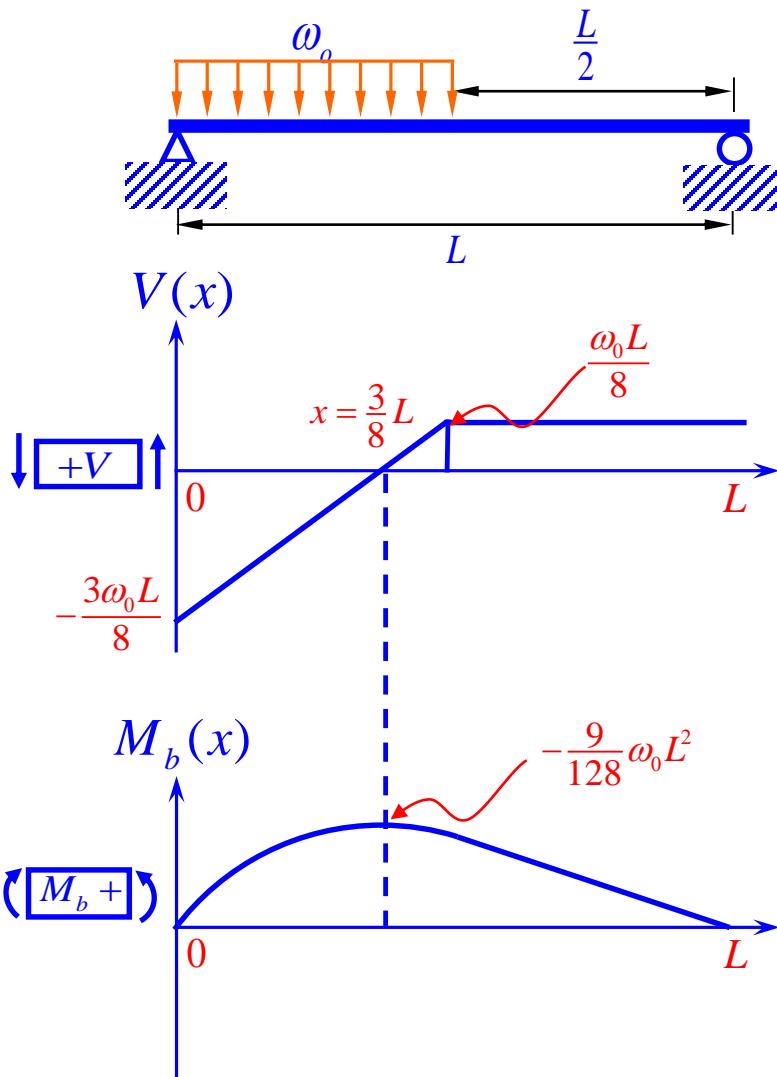
$$\begin{aligned}\Delta s &= \sqrt{\Delta x^2 + \Delta v^2} \\ &= \Delta x \sqrt{1 + \left(\frac{\Delta v}{\Delta x}\right)^2}\end{aligned}$$

$$\frac{\Delta x}{\Delta s} = \frac{1}{\sqrt{1 + \left(\frac{\Delta v}{\Delta x}\right)^2}}$$

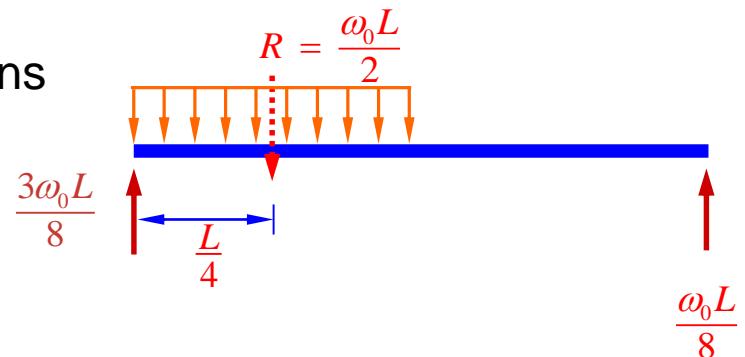
$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + v'^2(x)}}$$



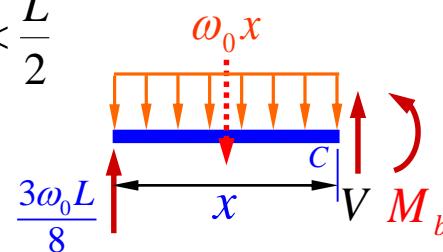
Example – Find SFD and BMD by cut method



○ Reactions



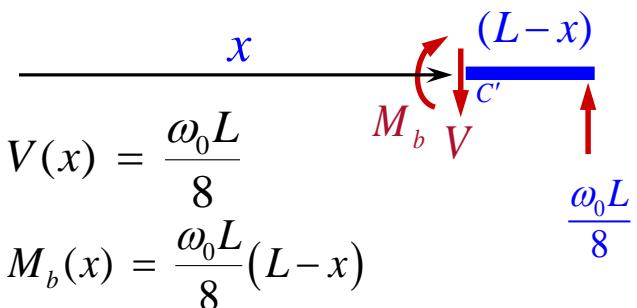
$$\textcircled{1} \quad 0 \leq x < \frac{L}{2}$$



$$\sum F_y = 0 ; \quad V(x) = \omega_0 x - \frac{3}{8}\omega_0 L$$

$$\sum M_C = 0 ; \quad M_b(x) = -\frac{\omega_0}{2}x^2 + \frac{3}{8}\omega_0 L x$$

$$\textcircled{2} \quad \frac{L}{2} \leq x \leq L$$

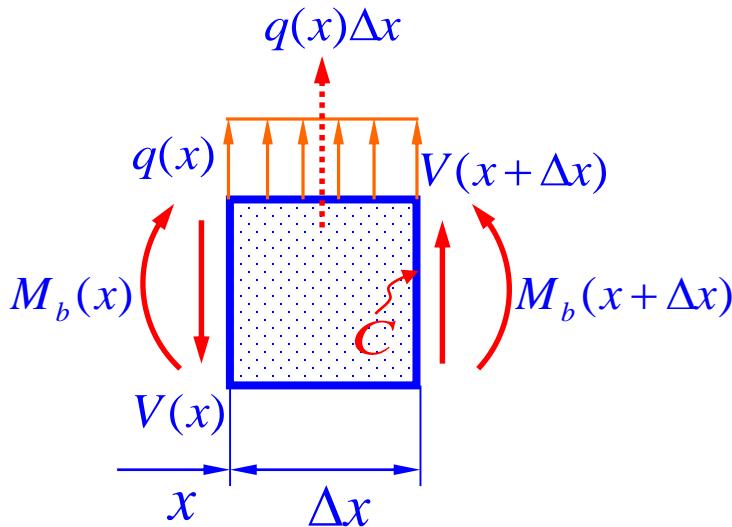


$$\sum F_y = 0 ; \quad V(x) = \frac{\omega_0 L}{8}$$

$$\sum M_{C'} = 0 ; \quad M_b(x) = \frac{\omega_0 L}{8}(L-x)$$



Relationship of $q(x)$ - $V(x)$ - $M_b(x)$



$$\sum F_y = 0; -V(x) + V(x + \Delta x) + q(x)\Delta x = 0$$

$$\lim_{\Delta x \rightarrow 0} \frac{V(x + \Delta x) - V(x)}{\Delta x} = -q(x)$$

$$\rightarrow \frac{dV(x)}{dx} = -q(x)$$

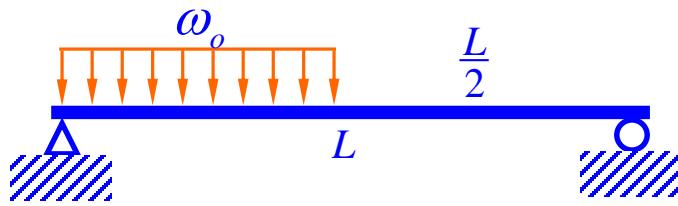
$$\sum M_C = 0; M_b(x + \Delta x) - M_b(x) + V(x)\Delta x - q(x)\Delta x \frac{\Delta x}{2} = 0$$

$$\lim_{\Delta x \rightarrow 0} \frac{M_b(x + \Delta x) - M_b(x)}{\Delta x} = -V(x) \quad \rightarrow \quad \frac{dM_b(x)}{dx} = -V(x)$$

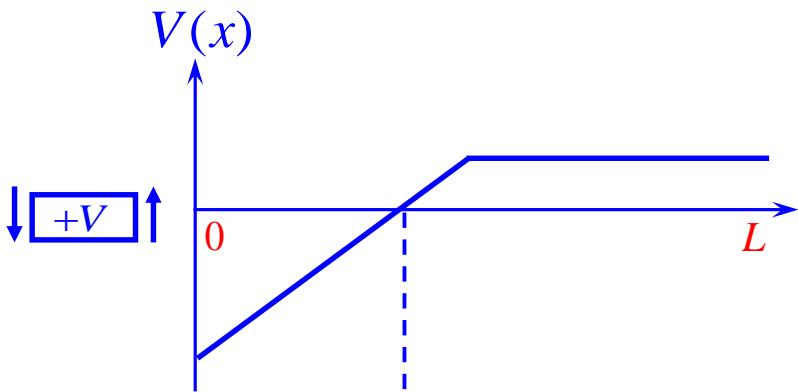
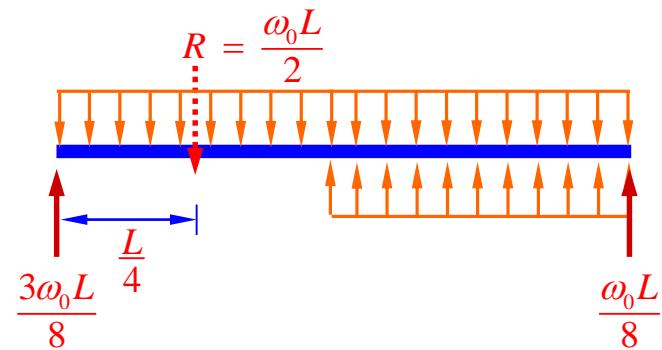
$$\left. \begin{aligned} \frac{dM_b(x)}{dx} &= -V(x), \quad M'_b(x) + V(x) = 0 \\ \frac{dV(x)}{dx} &= -q(x), \quad V'(x) + q(x) = 0 \end{aligned} \right\} M''_b(x) = q(x)$$



Example – Find SFD and BMD using DE

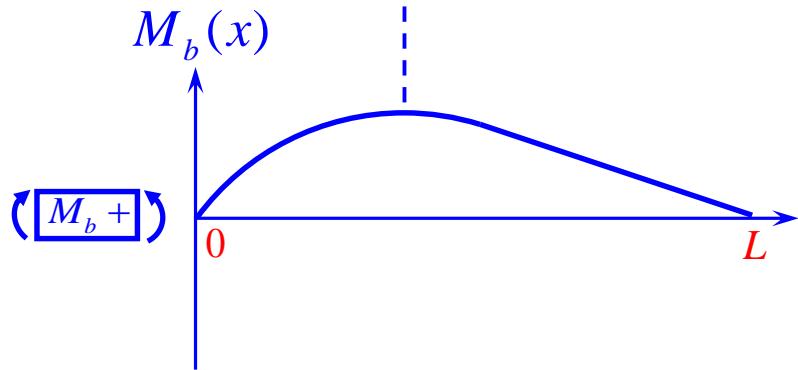


○ Reactions



○ Load intensity function

$$q(x) = \frac{3\omega_0 L}{8} < x >_{-1} - \omega_0 < x >^0 + \omega_0 < x - \frac{L}{2} >^0$$



○ Shear force and bending moment diagram

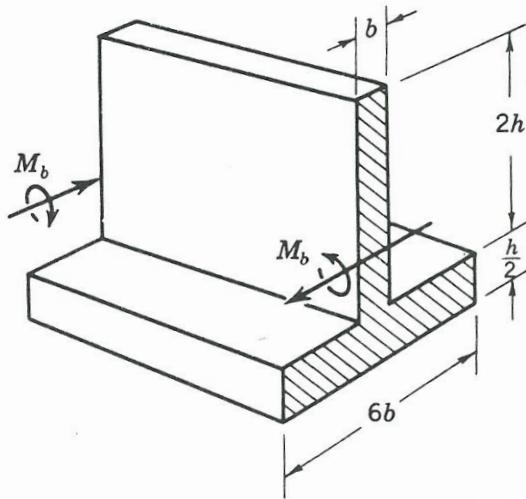
$$V(x) = -\frac{3\omega_0 L}{8} < x >^0 + \omega_0 < x >^1 - \omega_0 < x - \frac{L}{2} >^1$$

$$M_b(x) = \frac{3\omega_0 L}{8} < x >^1 - \frac{1}{2} \omega_0 < x >^2 + \frac{1}{2} \omega_0 < x - \frac{L}{2} >^2$$

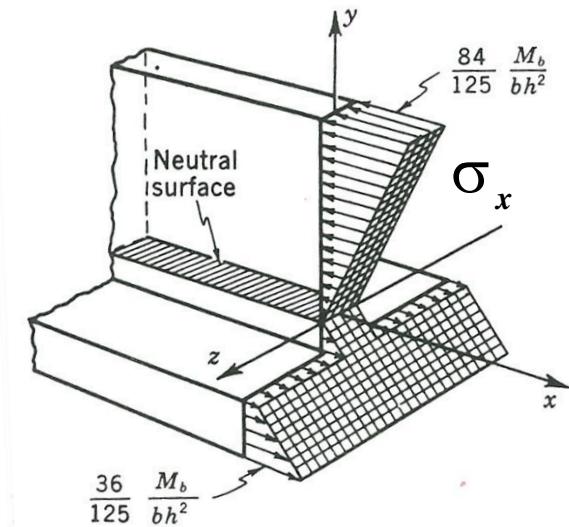


Distribution of internal force

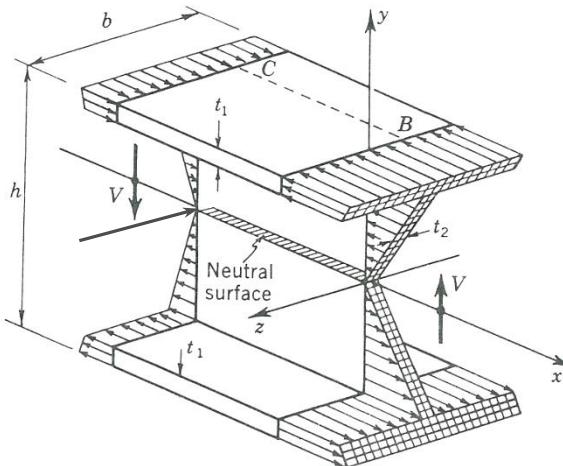
◎ Bending moment and bending stress



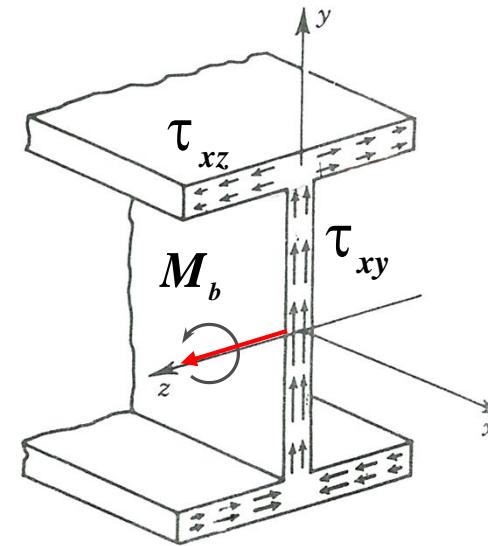
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◎ Shear force and shear stress



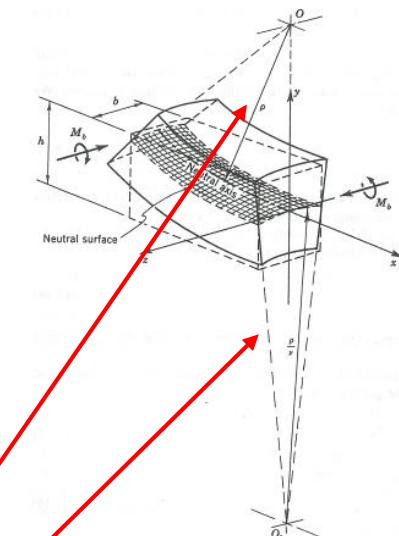
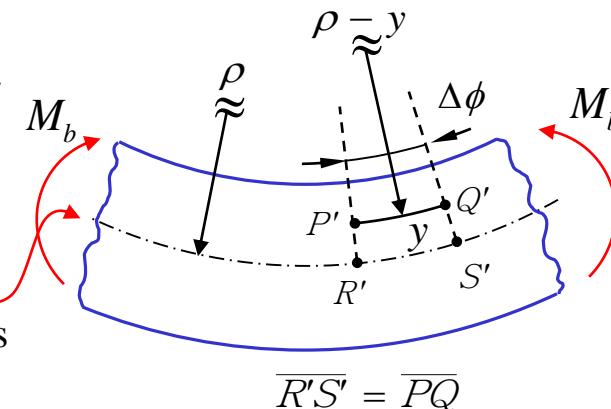
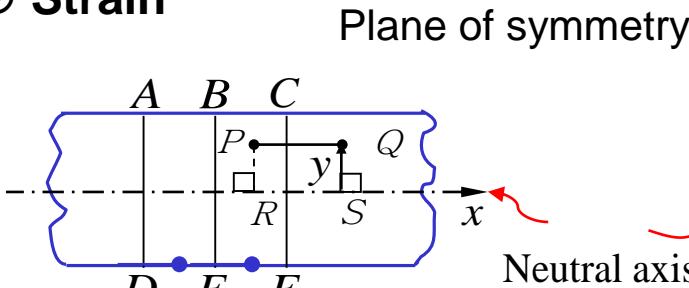
=





Pure bending – deflection curve-strain relation

○ Strain



$$\circ \quad \varepsilon_{xx} = \varepsilon_x = \frac{\overline{P'Q'} - \overline{PQ}}{\overline{PQ}} = \frac{(\rho - y)\Delta\phi - \rho\Delta\phi}{\rho\Delta\phi} = -\frac{y}{\rho} = -y v''(x)$$

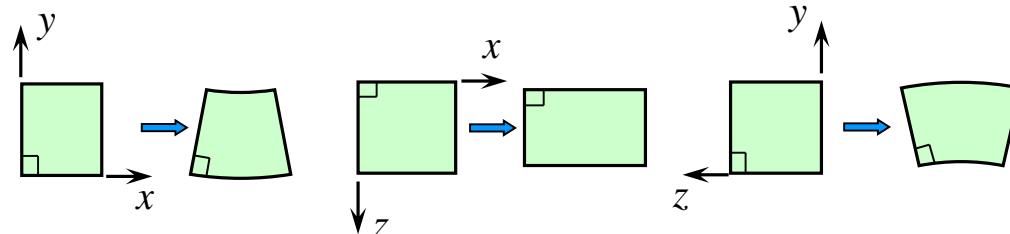
$$\circ \quad \varepsilon_y = \varepsilon_z = -v\varepsilon_x = \frac{vy}{\rho}$$

$$\circ \quad \gamma_{xy} = 0, \quad \gamma_{yz} = 0, \quad \gamma_{zx} = 0$$

$$\varepsilon_{zz} = \varepsilon_z = -\frac{y}{(-\rho/v)}$$

Anticlastic curvature

$$\begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\gamma_{xy} = 0$$

$$\gamma_{zx} = 0$$

$$\gamma_{yz} = 0$$

$$\begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & -v\varepsilon_x & 0 \\ 0 & 0 & -v\varepsilon_x \end{pmatrix}$$



Pure bending - stress-strain relation

- Hooke's law of an isotropic material

$$\cdot \varepsilon_{xx} = \frac{1}{E} [\sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz})]$$

$$\cdot \varepsilon_{yy} = \frac{1}{E} [\sigma_{yy} - \nu (\sigma_{zz} + \sigma_{xx})]$$

$$\cdot \varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy})]$$

$$\cdot \varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{(1+\nu)}{E} \sigma_{xy} = \frac{1}{2G} \sigma_{xy}$$

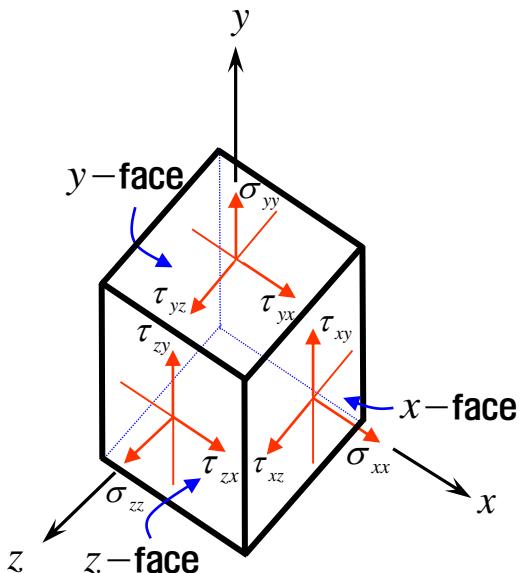
$$\cdot \varepsilon_{yz} = \frac{\gamma_{yz}}{2} = \frac{(1+\nu)}{E} \sigma_{yz} = \frac{1}{2G} \sigma_{yz}$$

$$\cdot \varepsilon_{zx} = \frac{\gamma_{zx}}{2} = \frac{(1+\nu)}{E} \sigma_{zx} = \frac{1}{2G} \sigma_{zx}$$

- Stress

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\tau_{ij} = \tau_{ji}$$

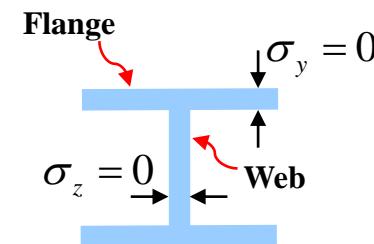


- Application of Hooke's law

- Assumption: $\sigma_y = 0, \sigma_z = 0$

- $\sigma_x = E\varepsilon_x = -E \frac{y}{\rho}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\nu \varepsilon_{xx}$

- $\sigma_{xy} = 0, \sigma_{yz} = 0, \sigma_{zx} = 0 (\tau_{xy} = 0, \tau_{yz} = 0, \tau_{zx} = 0)$

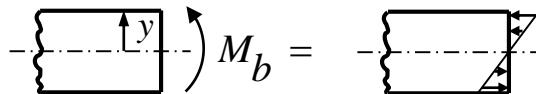


$$\begin{aligned} \varepsilon_{xx} &= \varepsilon_x \\ \varepsilon_{yy} &= \varepsilon_y \\ \varepsilon_{zz} &= \varepsilon_z \\ \sigma_{xx} &= \sigma_x \\ \sigma_{yy} &= \sigma_y \\ \sigma_{zz} &= \sigma_z \end{aligned}$$



Pure bending – Force equilibrium

- Application of requirement of force equilibrium



$$\sum F_x = 0 ; \int_A \sigma_x dA = 0$$

If $E = \text{constant}$

$$\sigma_x = -\frac{Ey}{\rho}, \quad \rho \neq \rho(A)$$

$$\frac{1}{\rho} \int_A E y dA = 0 \quad \downarrow \quad \int_A y dA = 0$$

$$\sum M_z = 0 ; \quad - \int_A y \sigma_x dA = M_b$$

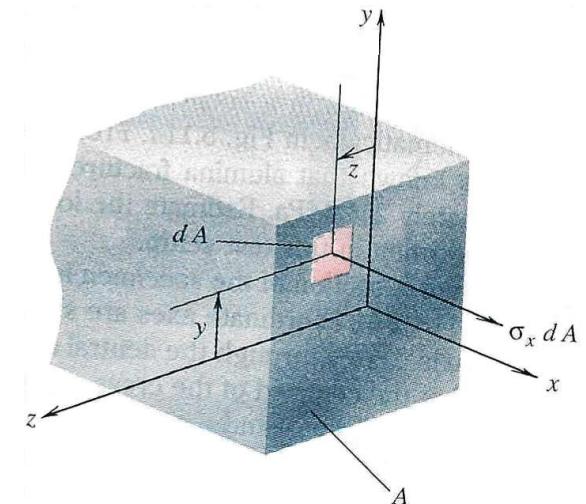
$\sigma_x = -\frac{Ey}{\rho}, \quad \rho \neq \rho(A)$

$$\frac{1}{\rho} \int_A E y^2 dA = M_b \quad \downarrow$$

If $E = \text{constant}$

$$\sum M_y = 0 ; \quad - \int_A z \sigma_x dA = \frac{E}{\rho} \int_A yz dA = 0$$

Automatically satisfied
for symmetric beam



• EI_{zz} : Flexural rigidity

$$\frac{EI_{zz}}{\rho} = M_b$$

$$I_{zz} \equiv \int_A y^2 dA$$

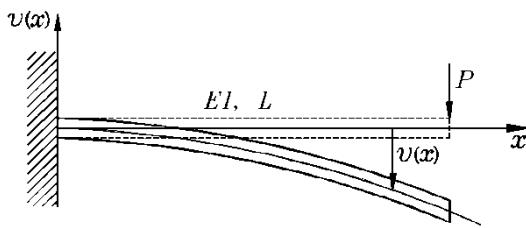
$$\sigma_x = -\frac{M_b y}{I_{zz}}$$

2nd moment of inertia
of the cross-section

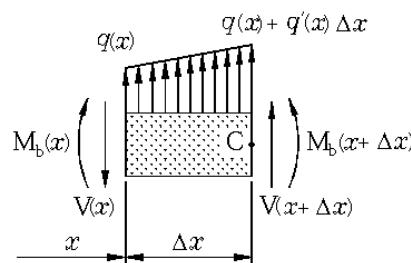


Summary of pure bending beam theory

- $q(x) = v(x) = M_b(x)$



$$\begin{cases} V(+), \\ M_b(+) \end{cases}$$



a) Positive V and M_b

b) Free-body diagram

c) Definition of coordinate system

- Curvature-deflection curve : $v''(x) = \frac{1}{\rho} = \kappa$

- Curvature-strain : $\varepsilon_{xx} = -\frac{y}{\rho}$

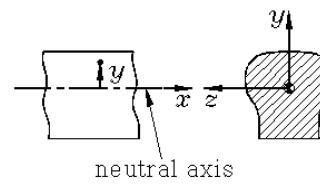
- Constitutive law : $\sigma_{xx} = -E\varepsilon_{xx} = -E \frac{y}{\rho}$

- Axial force : $\int_A \sigma_{xx} dA = 0 \rightarrow \int_A y dA = 0$

- Bending moment : $-\int_A \sigma_{xx} y dA = M_b \rightarrow \frac{1}{\rho} = \frac{M_b}{EI} = v''(x)$

$$EI v''(x) = M_b(x)$$

$$(EI v''(x))'' = M_b'' = q(x)$$



$$\sigma_{xx} = -\frac{M_b y}{I_{zz}}$$

• Boundary conditions

- Geometric : $v(0) = 0, v'(0) = 0$

$$v(L) = 0, v'(L) = 0$$

- Mechanical : $V(0) = P, M_b(0) = M, V(L) = P, \dots$

$$M_b(0) = EI v''(0) = M \rightarrow v''(0) = \frac{M}{EI}$$

$$V = -M_b'(x) \rightarrow V(0) = -(EI v''(0))' \rightarrow v'''(0) = -\frac{P}{EI}$$

$$\boxed{\frac{dV(x)}{dx} = -q(x)}$$

$$\frac{d^2 M_b(x)}{dx^2} = q(x)$$

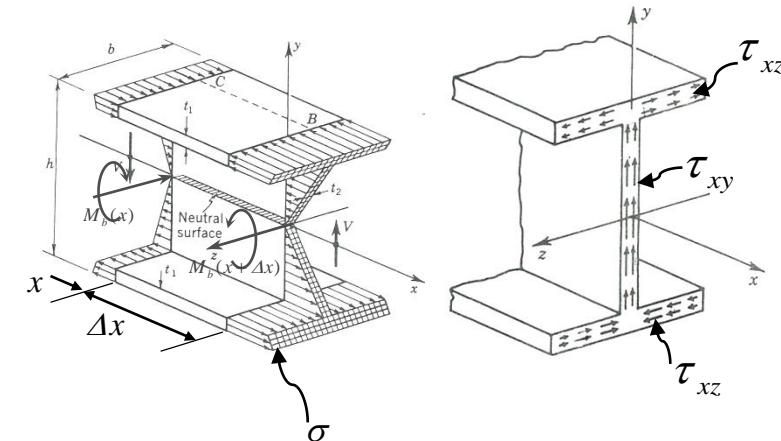
$$\boxed{\frac{dM_b(x)}{dx} = -V(x)}$$



Engineering beam theory

○ Purpose of engineering beam theory

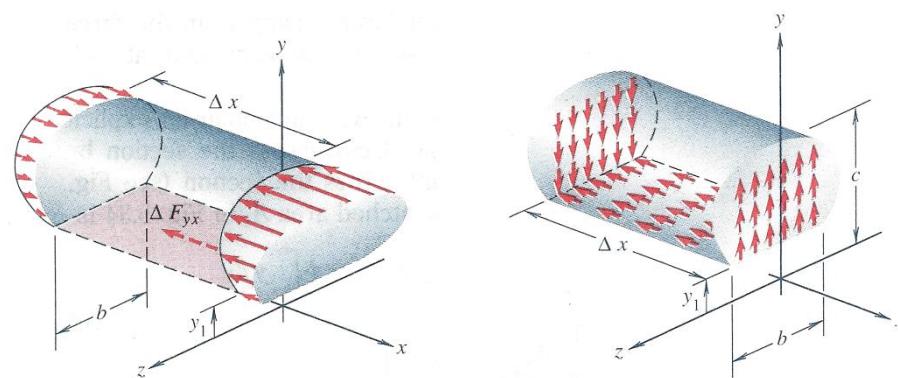
- When shear force exists,
 - bending moment varies from position to position.
 - shear stress as well as bending stress exist.
- Purpose of engineering beam theory is to calculate the shear stress



○ Assumption of engineering beam theory

- The following relationships obtained by the pure bending beam theory are valid even though shear force does not vanish.

$$\sigma_x = -\frac{M_b y}{I_{zz}}, \quad \frac{1}{\rho} = v''(x) = \frac{M_b}{EI_{zz}}$$



○ Results of engineering beam theory

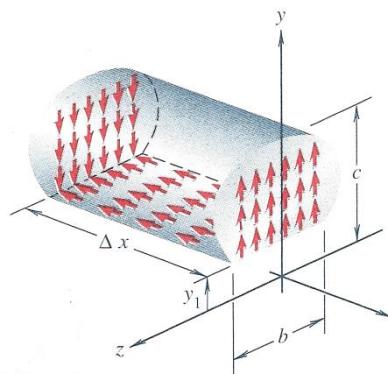
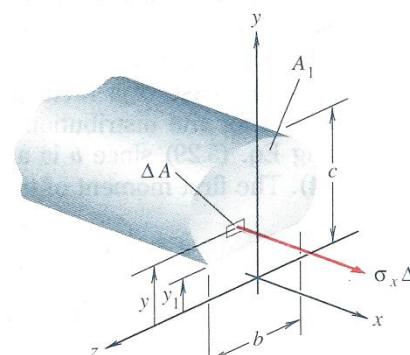
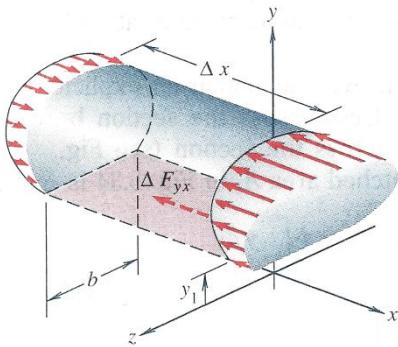
$$\tau_{yx} = \tau_{xy} = \frac{VQ}{bI_{zz}}$$

$$Q = \int_{A_l} y dA$$



Engineering beam theory – Shear stress

- Take a cut fraction having $-y$ face in the cut surface at $y = y_1$



Assumption: uniform cross-section

- A_l : Cross-sectional area defined by $y = y_1$

$$\sum F_x = \int_{A_l} \{(\sigma_x)_{x+\Delta x} - (\sigma_x)_x\} dA - \Delta F_{yx} = 0 \quad \text{---} \otimes$$

$$(\sigma_x)_{x+\Delta x} - (\sigma_x)_x = -\frac{(M_b(x+\Delta x) - M_b(x))y}{I_{zz}}$$

$$-\frac{M_b(x+\Delta x) - M_b(x)}{I_{zz} \Delta x} Q = \frac{\Delta F_{yx}}{\Delta x} \quad \leftarrow \quad Q = \int_{A_l} y dA$$

$$f \equiv \frac{dF_{yx}}{dx} = -\frac{dM_b}{dx} \frac{Q}{I_{zz}} = \frac{VQ}{I_{zz}} \quad \leftarrow \quad \text{Shear flow}$$

$$\Delta F_{yx} = \tau_{yx} b \Delta x \rightarrow \frac{dF_{yx}}{dx} = \tau_{yx} b$$

$$\boxed{\tau_{yx} = \tau_{xy} = \frac{VQ}{bI_{zz}}}$$

Assume: ΔF_{yx} is uniformly distributed over width b



Example 1 – Shear and bending stresses

- Given values

$$\sigma_a = 1000 \text{ psi}, b = h = 6", P = 1000 \text{ lb}$$

- Calculated maximum shear force and bending moment

$$M_{b(\max)} = \frac{LP}{4}, V_{\max} = \frac{P}{2}$$

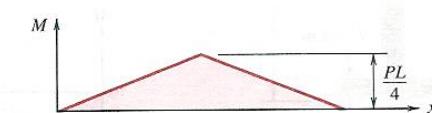
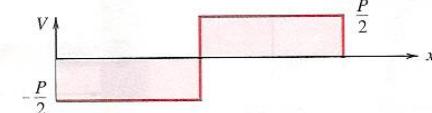
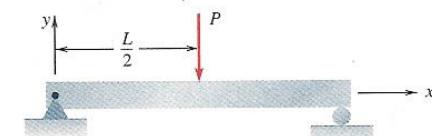
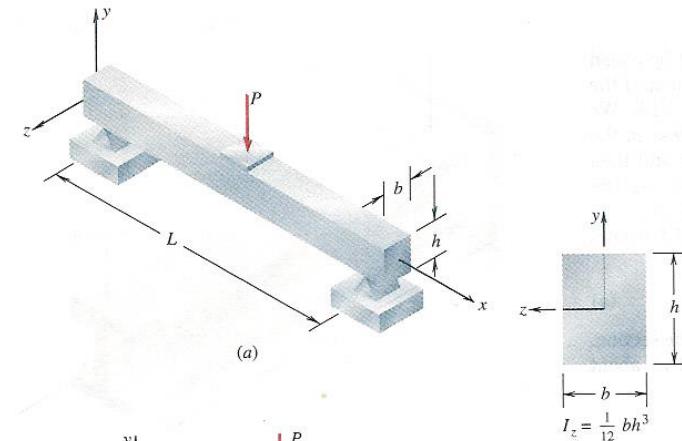
- Allowable maximum beam length L_{\max}

$$\sigma_{x(\max)} = -\frac{M_b y}{I_z} \Big|_{y=-h/2} = -\frac{LP/4 \times (-h/2)}{bh^3/12} = \frac{3LP}{2bh^2}$$

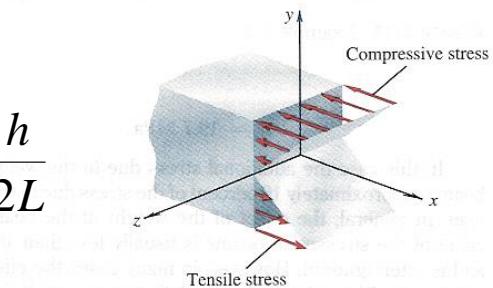
$$\sigma_{x(\max)} = \sigma_a \rightarrow L_{\max} = \frac{2bh^2\sigma_a}{3P} = \frac{2 \times 6^3 \text{ in}^3 \times 1000 \text{ lb}}{3000 \text{ lb} \cdot \text{in}^2} = 144 \text{ in}$$

- Ratio of maximum shear stress to bending stress

$$\tau_{xy(\max)} = \frac{VQ}{bI} = \frac{P/2 \times bh/2 \times h/4}{b \times bh^3/12} = \frac{3P}{4bh} \rightarrow \frac{\tau_{xy(\max)}}{\sigma_{x(\max)}} = \frac{\frac{3P}{4bh}}{\frac{3LP}{2bh^2}} = \frac{h}{2L}$$

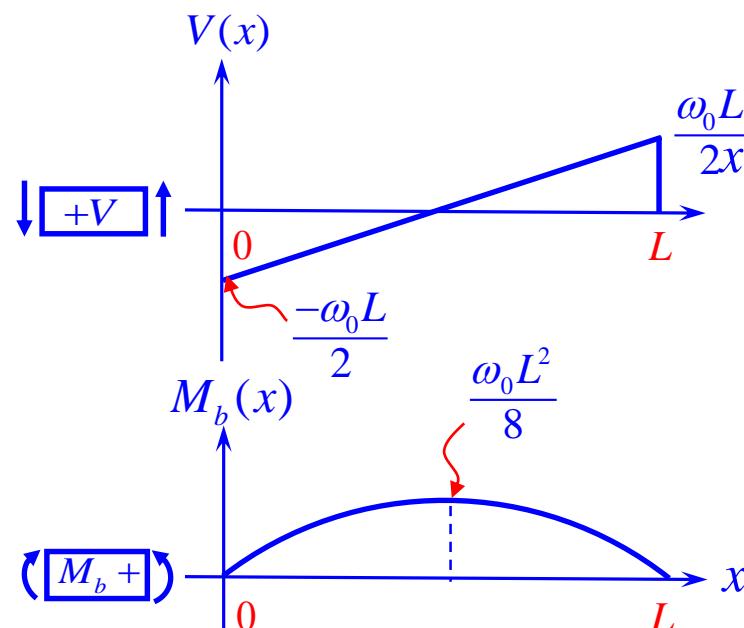
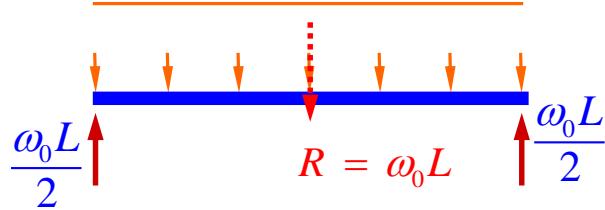
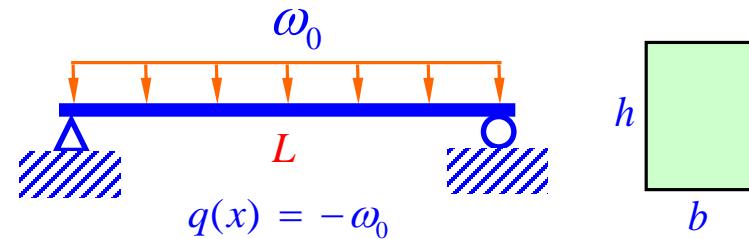


(b)





Example 2 – Ratio of shear to bending stresses



$$q(x) = \frac{\omega_o L}{2} < x >_{-1} - \omega_o < x >^0 + \frac{\omega_o L}{2} < x - L >_{-1}$$

$$V(x) = -\frac{\omega_o L}{2} < x >^0 + \omega_o < x >^1 - \frac{\omega_o L}{2} < x - L >^0$$

$$M_b(x) = \frac{\omega_o L}{2} < x >^1 - \frac{1}{2} \omega_o < x >^2 + \frac{\omega_o L}{2} < x - L >^1$$

$$M_b\left(\frac{L}{2}\right) = \frac{\omega_o L}{2} \times \frac{L}{2} - \frac{1}{2} \omega_o \times \frac{L^2}{4} = \frac{\omega_0 L^2}{8}$$

$$\tau_{xy\text{ max}} = \frac{\frac{\omega_0 L}{2} \times b \times \frac{h}{2} \times \frac{h}{4}}{b \times \frac{1}{12} b h^3} = \frac{3\omega_0 L}{4bh}$$

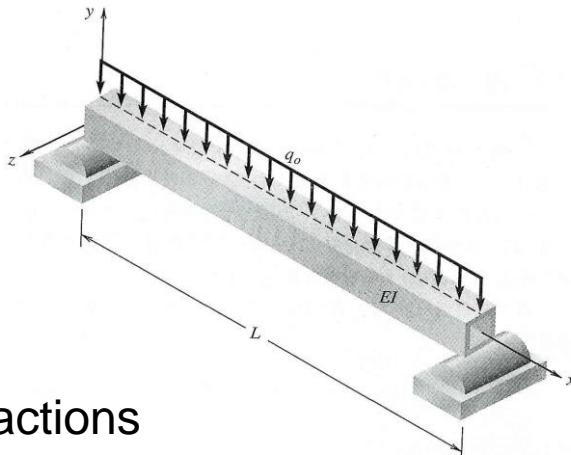
$$\sigma_{xy\text{ max}} = -\frac{\frac{\omega_0 L^2}{8} \times \left(-\frac{h}{2}\right)}{\frac{1}{12} b h^3} = \frac{3\omega_0 L^2}{4bh^2}$$

$$\therefore \frac{\tau_{xy\text{ max}}}{\sigma_{xy\text{ max}}} = \frac{h}{L}$$

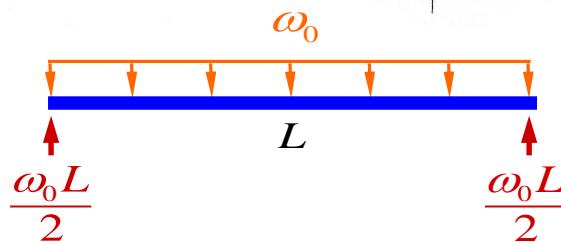


Example 3 – Beam deflection

○ Problem



○ Reactions



○ Boundary conditions

$$\circ \begin{cases} v(0) = 0 \\ v(L) = 0 \end{cases}$$

$$\circ \begin{cases} M_b(0) = EIv''(0) = 0 \\ M_b(L) = EIv''(L) = 0 \end{cases}$$

○ $v(x)$?

- $q(x) = \frac{\omega_0 L}{2} \langle x \rangle_{-1} - \omega_0 \langle x \rangle^0$
- $EIv^{(4)}(x) = q(x) = \frac{\omega_0 L}{2} \langle x \rangle_{-1} - \omega_0 \langle x \rangle^0$
- $EIv'''(x) = \frac{\omega_0 L}{2} \langle x \rangle^0 - \omega_0 \langle x \rangle^1 + C_A$
- $EIv''(x) = \frac{\omega_0 L}{2} \langle x \rangle^1 - \frac{\omega_0}{2} \langle x \rangle^2 + C_A x + C_B$
- $EIv'(x) = \frac{\omega_0 L}{4} \langle x \rangle^2 - \frac{\omega_0}{6} \langle x \rangle^3 + C_1$
- $EIv(x) = \frac{\omega_0 L}{12} \langle x \rangle^3 - \frac{\omega_0}{24} \langle x \rangle^4 + C_1 x + C_2$

○ Determination of C_1, C_2

- $v(0) = 0 \rightarrow C_2 = 0$
- $v(L) = 0 \rightarrow \frac{\omega_0}{12} L^4 - \frac{\omega_0}{24} L^4 + C_1 L = 0 \rightarrow C_1 = -\frac{\omega_0 L^3}{24}$

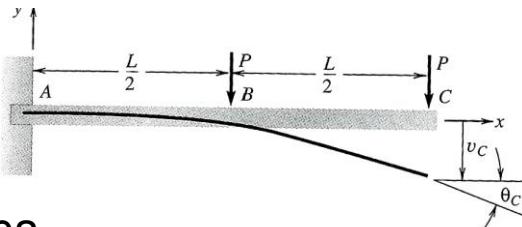
$$v(x) = \frac{1}{EI} \left(\frac{\omega_0 L}{12} \langle x \rangle^3 - \frac{\omega_0}{24} \langle x \rangle^4 - \frac{\omega_0 L^3}{24} x \right)$$

$$= \frac{\omega_0}{24EI} (2Lx^3 - x^4 - L^3 x)$$

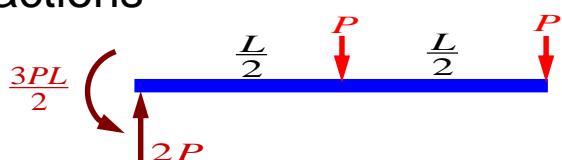


Principle of superposition

○ Problem description



○ Reactions

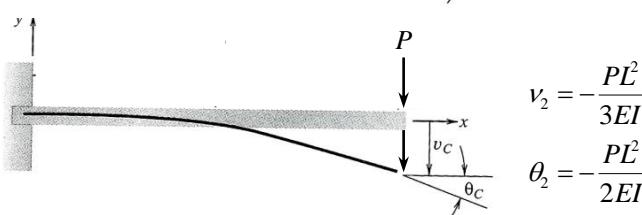
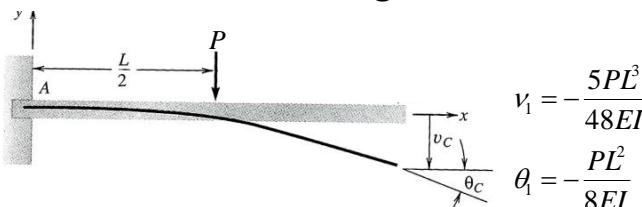


○ Boundary conditions

- $v(0) = 0, v'(0) = 0$

- $v''(L) = 0, v'''(L) = 0$

○ Deflection and angle due to each force



○ $M_b(x)$

- $q(x) = \frac{3}{2}PL\langle x \rangle_{-2} + 2P\langle x \rangle_{-1} - P\langle x - \frac{L}{2} \rangle_{-1}$

- $M_b(x) = -\frac{3}{2}PL + 2P\langle x \rangle^1 - P\langle x - \frac{L}{2} \rangle^1$

○ $v(x), v(L), \theta(L) = v'(L)$

- $EIv''(x) = M_b(x)$

- $EIv'(x) = -\frac{3}{2}PLx + Px^2 - \frac{P}{2}\langle x - \frac{L}{2} \rangle^2 + \mathcal{C}_1 \quad (\because v'(0) = 0)$

- $EIv(x) = -\frac{3}{4}PLx^2 + \frac{P}{3}x^3 - \frac{P}{6}\langle x - \frac{L}{2} \rangle^3 + \mathcal{C}_2 \quad (\because v(0) = 0)$

- $v(L) = \frac{PL^3}{EI} \left(-\frac{3}{4} + \frac{1}{3} - \frac{1}{6 \times 8} \right) = -\frac{7PL^3}{16EI}$

- $\theta(L) = -\frac{5PL^2}{8EI}$

○ Application of principle of superposition

$$v = v_1 + v_2 = -\frac{PL^2}{EI} \left(\frac{1}{3} + \frac{5}{48} \right) = -\frac{7PL^2}{16EI}$$

$$\theta = \theta_1 + \theta_2 = -\frac{5PL^2}{8EI}$$



3.9 Buckling



Governing equation of buckling of column

- Similarity and difference between beam and column
- Similarity: Bending moment and shear force are exerted and pure bending theory is applied.
- In beam theory, axial force (P) is neglected in calculating bending moment.
However, in column theory, axial force should be considered in calculating bending moment.

- Relationship of M_b , V , P , and $q(x)$

$$\sum F_y = 0 ; q(x)\Delta x + \frac{dV}{dx}\Delta x = 0 \rightarrow \frac{dV}{dx} + q(x) = 0 \rightarrow \frac{dV}{dx} = -q(x)$$

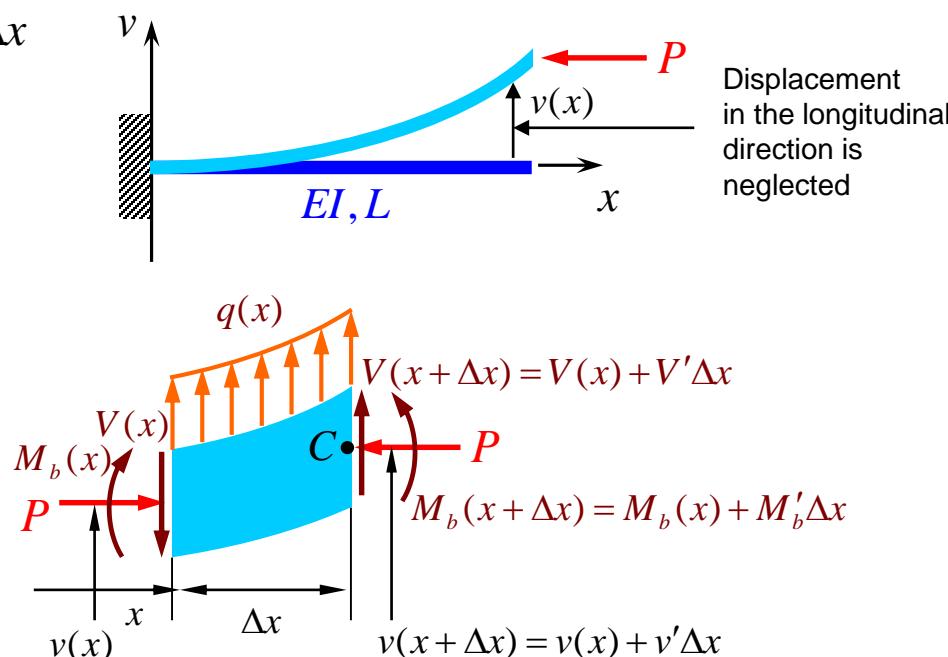
$$\sum M_C = 0 ; M_b(x + \Delta x) - M_b(x) + V(x)\Delta x$$

$$-\frac{q(x)}{2}\Delta x^2 + P(v(x + \Delta x) - v(x)) = 0$$

$$\rightarrow \frac{dM_b}{dx} + V + P\frac{dv}{dx} = 0$$

$$\rightarrow \frac{d^2M_b}{dx^2} + \frac{dV}{dx} + \frac{d}{dx}(P\frac{dv}{dx}) = 0$$

$$\rightarrow \frac{d^2M_b}{dx^2} + \frac{d}{dx}(P\frac{dv}{dx}) = q(x)$$





Governing equation of buckling of column

◎ Assume: Pure bending theory is applied.

$$\kappa = \frac{1}{\rho} \equiv v''(x)$$

$$\kappa = \frac{1}{\rho} = \frac{M_b}{EI}$$

$$\int_A y dA = 0$$

$$\sigma_x = -E \frac{y}{\rho}$$

$$\varepsilon_x = -\frac{y}{\rho}$$

Pure bending

◎ Governing equations

$$\frac{d^2}{dx^2}(M_b) + \frac{d}{dx}(P \frac{dv}{dx}) = q(x)$$

$$\begin{aligned} \frac{dV}{dx} + q(x) &= 0 \\ \frac{dM_b}{dx} + V + P \frac{dv}{dx} &= 0 \end{aligned}$$

$$\frac{d^2}{dx^2}(EI \frac{d^2v}{dx^2}) + \frac{d}{dx}(P \frac{dv}{dx}) = q(x)$$

$$\begin{aligned} EI &= \text{일정}, P = \text{일정} \\ q(x) &= 0 \end{aligned}$$

$$v^{(4)}(x) + \beta^2 v''(x) = 0, \quad \beta^2 = \frac{P}{EI}$$

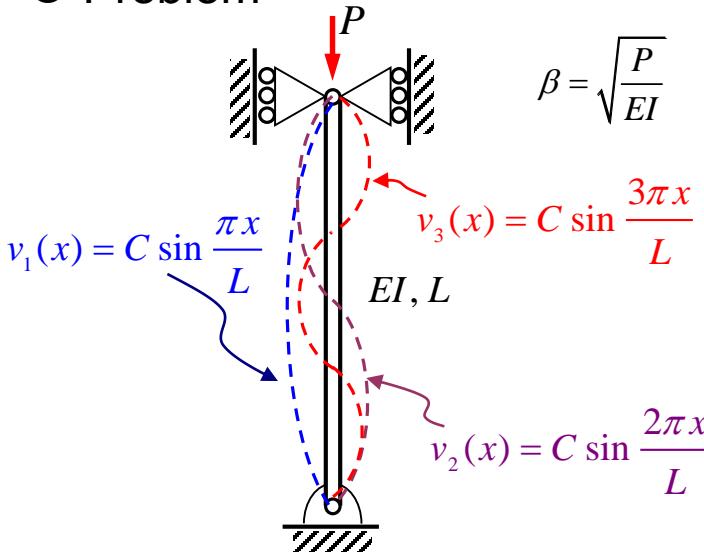
$$\begin{aligned} s^4 + \beta^2 s^2 &= 0 \rightarrow s = 0(\text{중근}), s = \pm \beta i \\ \therefore v(x) &= C_1 + C_2 x + C_3 \sin \beta x + C_4 \cos \beta x \end{aligned}$$

C_1, C_2, C_3, C_4 are calculated by BCs.



Example – buckling loads and modes

○ Problem



○ GE and homogeneous solution

$$v^{(4)}(x) + \frac{P}{EI} v''(x) = 0, \quad \beta^2 = \frac{P}{EI}$$

$$v(x) = C_1 + C_2 x + C_3 \sin \beta x + C_4 \cos \beta x$$

○ BCs

$$v(0) = 0, \quad v(L) = 0$$

$$M_b(0) = 0 \rightarrow v''(0) = 0$$

$$M_b(L) = 0 \rightarrow v''(L) = 0$$

○ Solving

$$v''(x) = -C_3 \beta^2 \sin \beta x - C_4 \beta^2 \cos \beta x \quad (1)$$

$$v(0) = C_1 + C_4 = 0 \quad (2)$$

$$v(L) = C_1 + C_2 L + C_3 \sin \beta L + C_4 \cos \beta L = 0 \quad (3)$$

$$v''(0) = -C_4 \beta^2 = 0 \quad (4)$$

$$v''(L) = -C_3 \beta^2 \sin \beta L - C_4 \beta^2 \cos \beta L = 0 \quad (5)$$

○ Because $\beta \neq 0$, $C_4 = 0$ from Eq. ④. $C_1 = 0$ from Eq. ①. In order to obtain non-trivial solution, $\sin \beta L = 0$.

$$v(x) = C_3 \sin \beta x$$

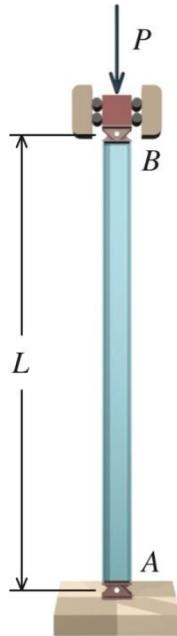
$$\sin \beta L = 0 \rightarrow \beta L = n\pi \rightarrow \beta = n\pi / L (n = 1, 2, \dots, \infty)$$

$$\sqrt{\frac{P_n}{EI}} L = n\pi \rightarrow P_n = n^2 \pi^2 \frac{EI}{L^2} \rightarrow v_n(x) = C \sin \frac{n\pi x}{L}$$

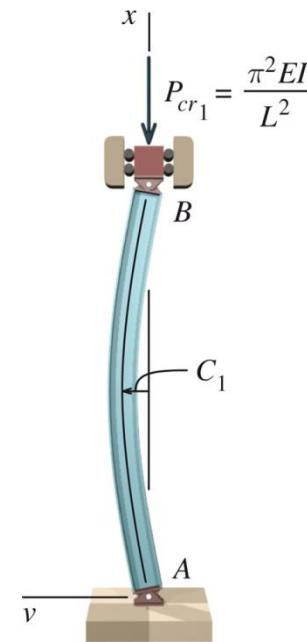
$$\sqrt{\frac{P_{cr}}{EI}} L = \pi \rightarrow P_{cr} = \pi^2 \frac{EI}{L^2} \quad \text{Buckling load}$$



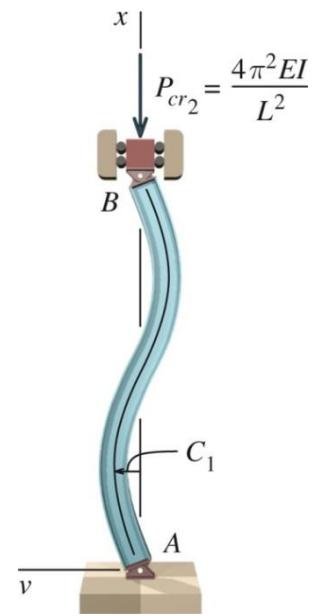
Mode of buckling



Non-buckled



Buckling mode 1
($n=1$)



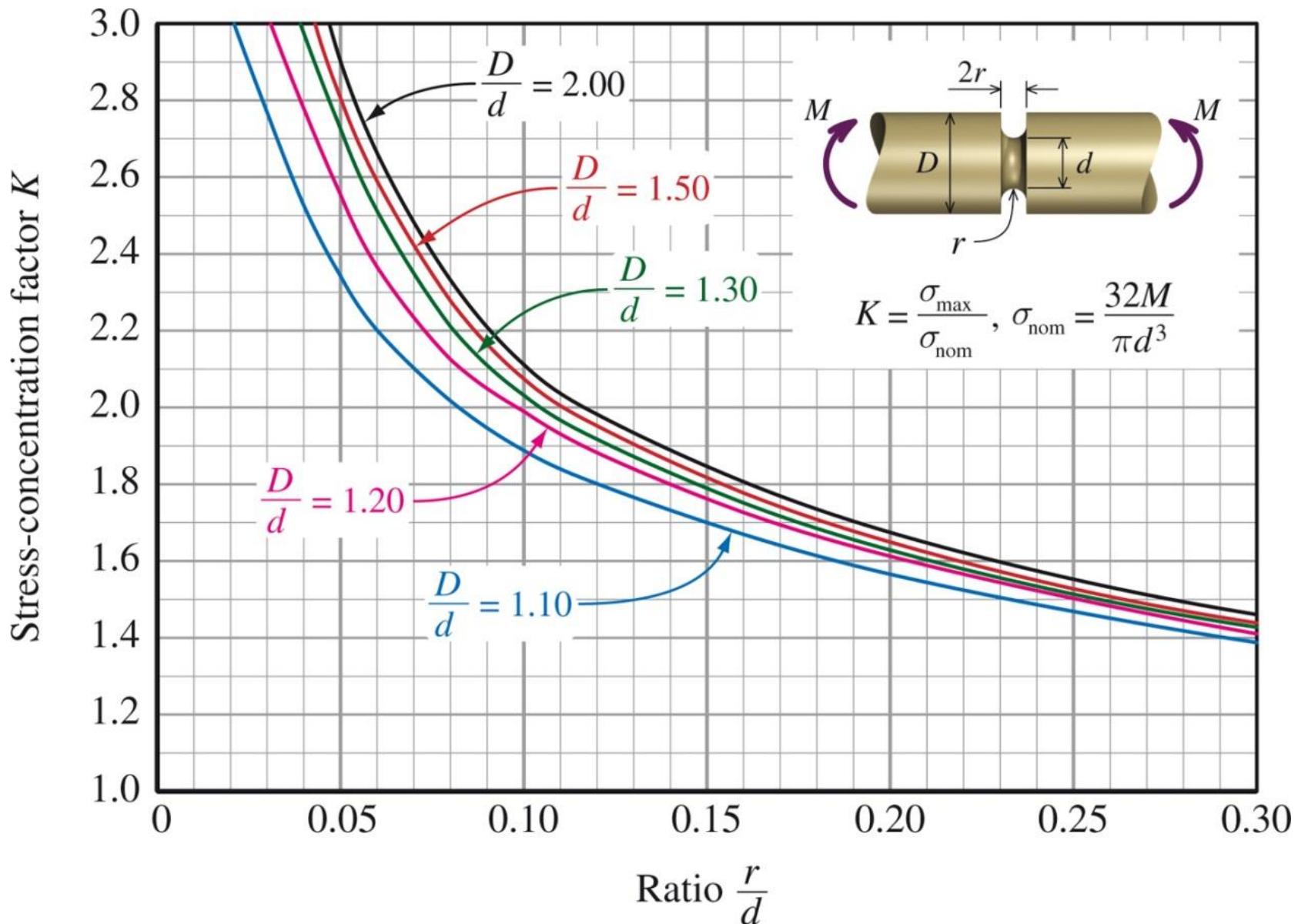
Buckling mode 2
($n=2$)



3.10 Special phenomenon



Stress concentration(SC) and SC factor K



$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}}, \quad \sigma_{\text{nom}} = \frac{32M}{\pi d^3}$$