

# **II. Mathematical Backgrounds**

**2.1 Vector**

**2.2 Matrix and Linear Algebra**

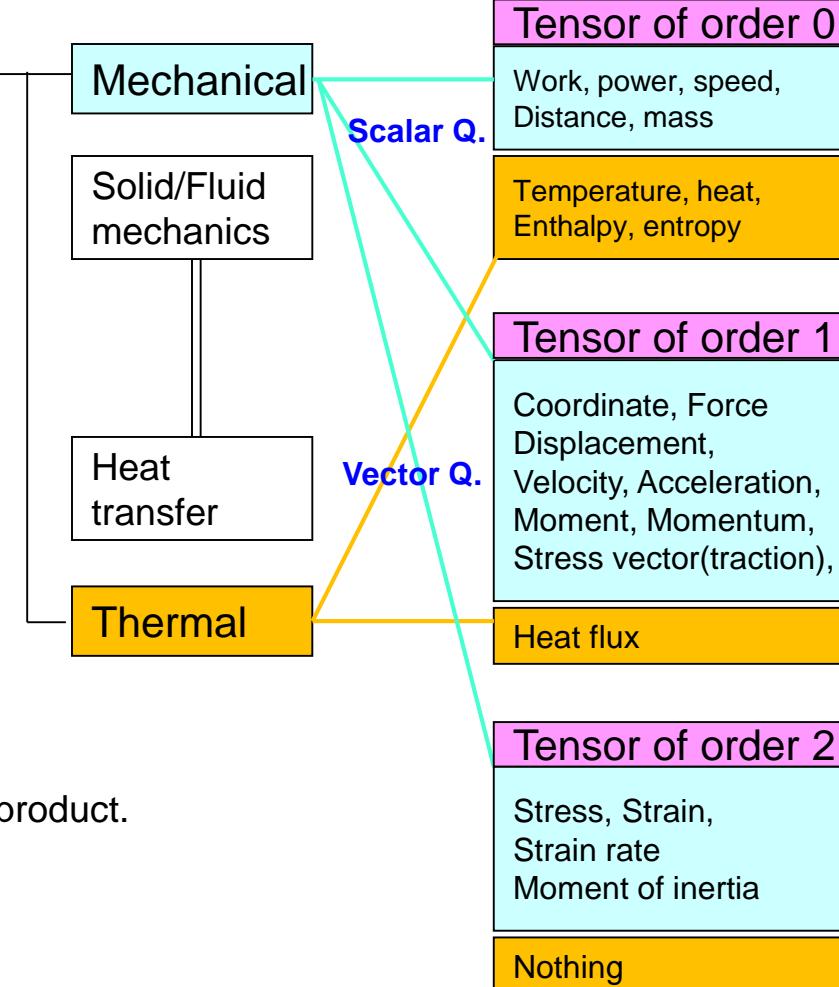
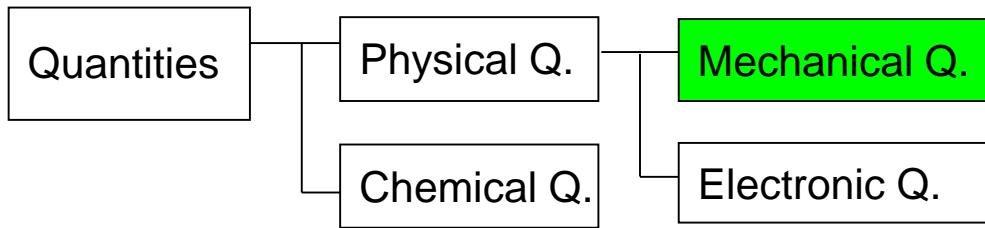
**2.3 Function and Differentiation**

**2.4 Integration**

# **2.1 Vector**



# Mechanical Quantities



## ❖ Tensor of order 0 (Scalar Q.)

- Scalar quantity has only magnitude of real number.
- Real number calculation ( $+$ ,  $-$ ,  $\times$ ,  $/$ ). Function.

## ❖ Tensor of order 1 (Vector Q.)

- Vector calculus is needed to learn vector mechanics.
- Addition, real number multiplication, inner product, cross product.
- Coordinate transformation

## ❖ Tensor of order 2

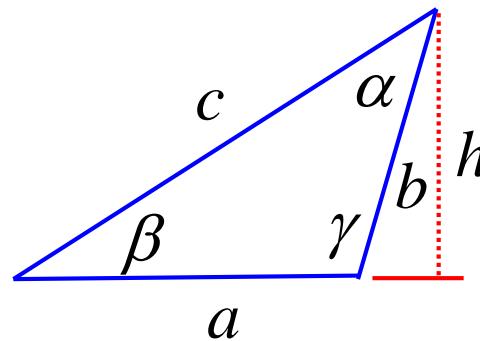
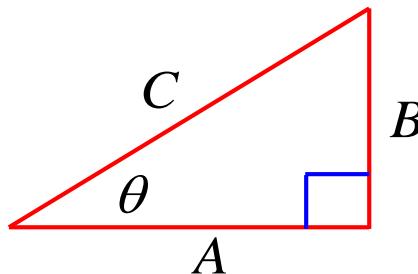
- Matrix algebra is needed to understand stress, strain, etc.
- Addition, real number multiplication, matrix multiplication
- Coordinate transformation. Eigenvalue problem.

**What is CAE?**



# Preliminary - Special functions

## ○ Trigonometric function



$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\sin \theta = \frac{B}{C}, \cos \theta = \frac{A}{C}, \tan \theta = \frac{B}{A}, \theta = \tan^{-1} \frac{B}{A}$$

$$\sin x = x - \frac{x^3}{3!} + \dots, \quad \cos x = 1 - \frac{x^2}{2!} + \dots$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{\cot \theta}$$

$$\sin 0^\circ = 0, \sin 30^\circ = 1/2, \sin 45^\circ = \sqrt{2}/2$$

$$\sin 60^\circ = \sqrt{3}/2, \sin 90^\circ = 1$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos 0^\circ = 1, \cos 30^\circ = \sqrt{3}/2, \cos 45^\circ = \sqrt{2}/2$$

$$\cos 60^\circ = 1/2, \cos 90^\circ = 0$$

$$\sin' \theta = \cos \theta, \cos' \theta = -\sin \theta, \tan' \theta = \sec^2 \theta$$

$$\sin(-\theta) = -\sin \theta, \cos(-\theta) = \cos \theta$$

$$y = \sin \theta \Leftrightarrow \theta = \sin^{-1} y = \arcsin y$$

## ○ Logarithmic function and exponential function

$$y = e^x, \quad y' = e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$y = \ln x, \quad y' = \frac{1}{x}$$

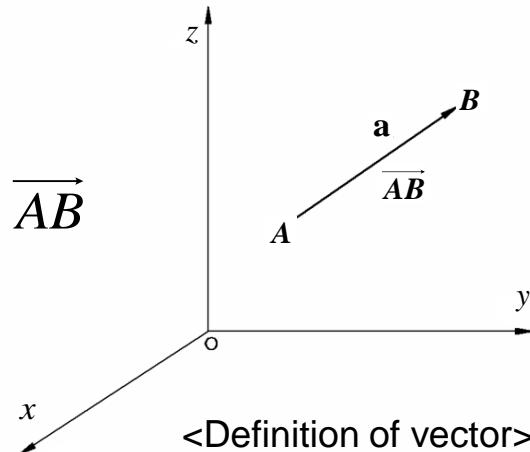


# Definition of vector

- Vector quantity: Vector quantity is defined as the quantity which has direction as well as magnitude. Vector is a mathematical description of vector quantity.

## ○ Description of vector

- Vector:  $\mathbf{a}$ ,  $\mathbf{x}$ ,  $\vec{a}$ ,  $\vec{x}$ ,  $\overrightarrow{AB}$
- Magnitude:  $|\mathbf{a}|$ ,  $|\vec{a}|$ ,  $a$



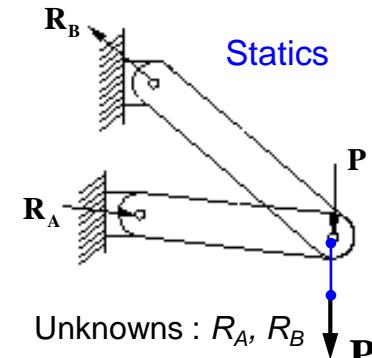
## ○ Factors of vector

- Essential factors (Mathematical requirements)
  - **Magnitude:** Distance between points  $A$  and  $B$
  - **Direction:** Direction of arrow directing from point  $A$  to  $B$

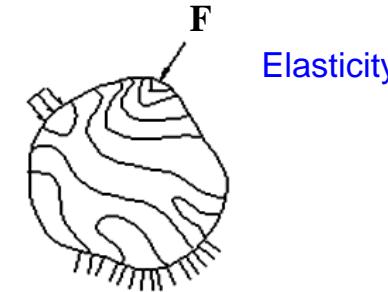
## ○ Selective factors

- **Point of action:** Point  $B$
- **Line of action:** Line passing points  $A$  and  $B$

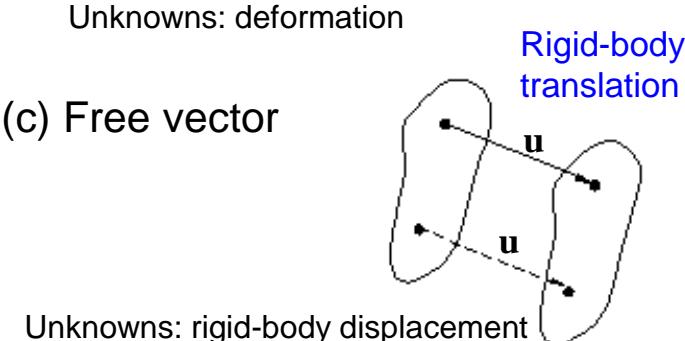
(a) Sliding vector



(b) Bound vector



(c) Free vector





# Mathematical description of vector

## ◎ Mathematical description: Component

### ○ On 2D plane

● Row vector:  $\mathbf{a} = [a_x, a_y]$  or  $[a_x \ a_y]$

● Column vector:  $\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x, a_y]^T$

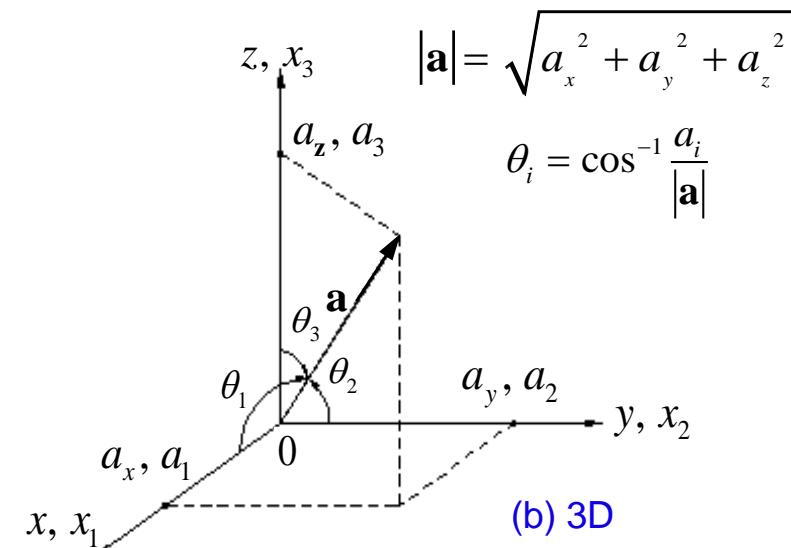
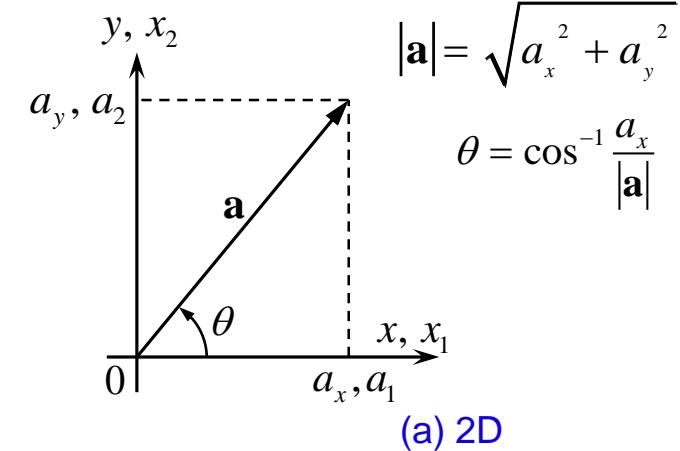
### ○ In 3D space

● Row vector:  $\mathbf{a} = [a_x, a_y, a_z]$

● Column vector:  $\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_x, a_y, a_z]^T$

✿ Components can make us calculate direction as well as magnitude of a vector.

Basically, a vector in mechanics means row vector

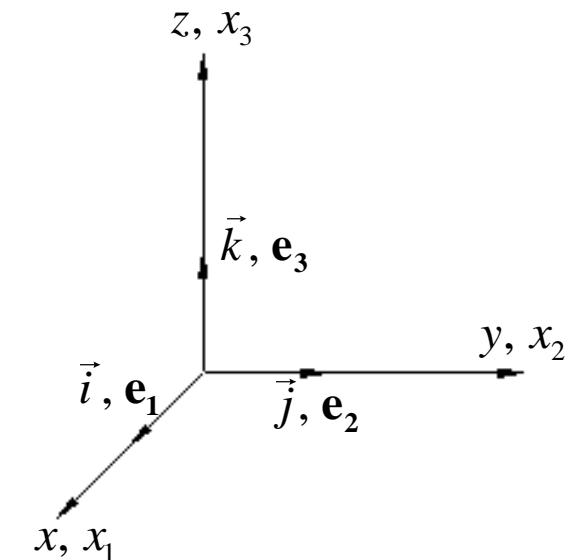


<Components in the rectangular coordinate system>



# Definition of terminology

- Zero vector:  $\vec{0} = \mathbf{0} = [0, 0, 0]^T$
- Magnitude of vector  $\mathbf{a}$ :  $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$       or       $\sqrt{a_1^2 + a_2^2 + a_3^2}$
- Direction of vector  $\mathbf{a}$ :  $\theta_i = \cos^{-1} \frac{a_i}{|\mathbf{a}|}$  ( $i=1, 2, 3$ )
- Directional cosine :  $[\cos\theta_1, \cos\theta_2, \cos\theta_3]^T$
- Unit vector: A vector with magnitude of unity



- $|\mathbf{u}| = 1$    or    $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$  ( $|\mathbf{a}| \neq 0$ )

- Unit basis vector :

- $\vec{i} = \mathbf{i} = [1, 0, 0]^T = \mathbf{e}_1$

- $\vec{j} = \mathbf{j} = [0, 1, 0]^T = \mathbf{e}_2$

- $\vec{k} = \mathbf{k} = [0, 0, 1]^T = \mathbf{e}_3$

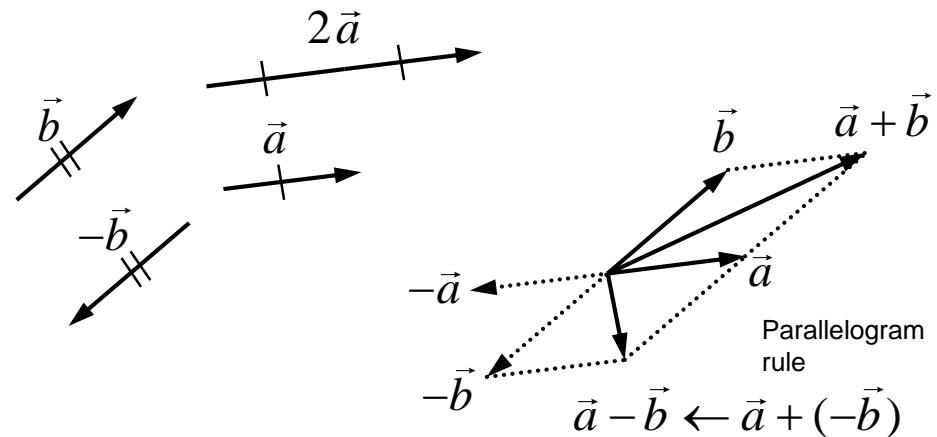
<x-y-z coordinate system  
and unit basis vector>



# Algebra of vector – vector addition

- Addition of vectors

$$\mathbf{a} + \mathbf{b} \equiv [a_1 + b_1, a_2 + b_2, a_3 + b_3]^T$$

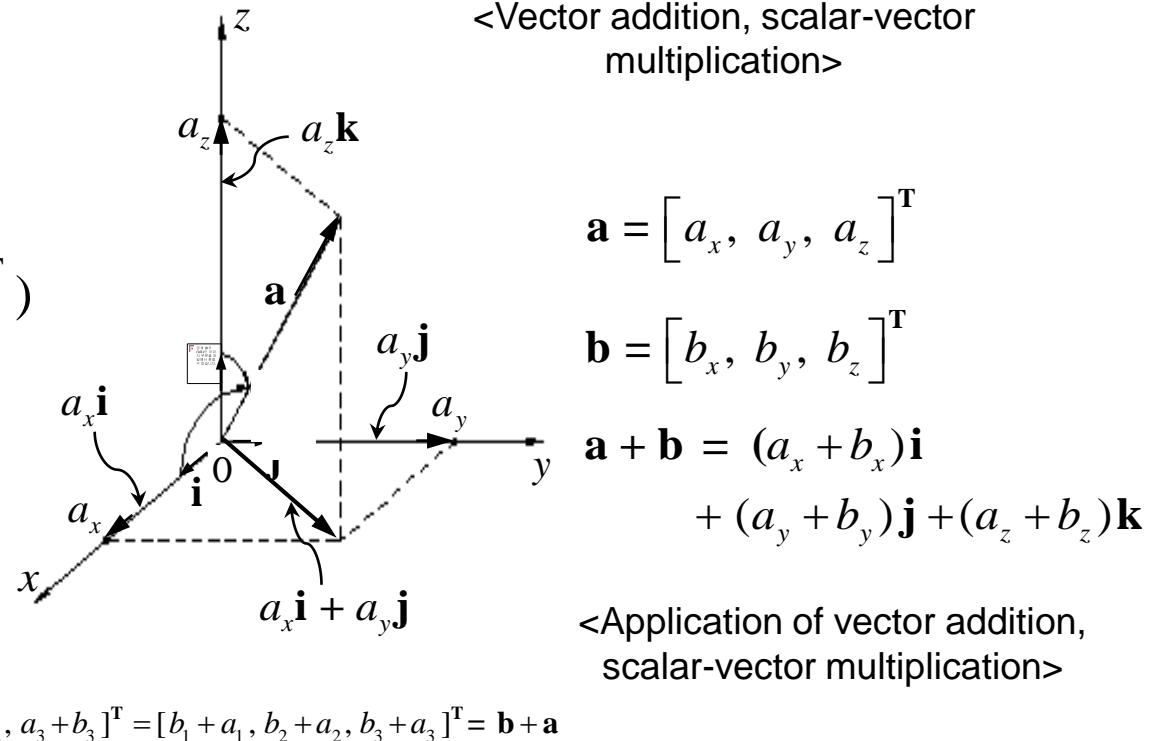


- Multiplication of a scalar and a vector

$$\alpha \mathbf{a} \equiv [\alpha a_1, \alpha a_2, \alpha a_3]^T$$

- Characteristics of vector addition and scalar-vector multiplication

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{0} + \mathbf{a} = \mathbf{a}$  ( $\mathbf{0} = [0 \ 0 \ 0]^T$ )
- $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
- $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
- $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$
- $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$
- $1\mathbf{a} = \mathbf{a}$





# Algebra of vector – Inner product

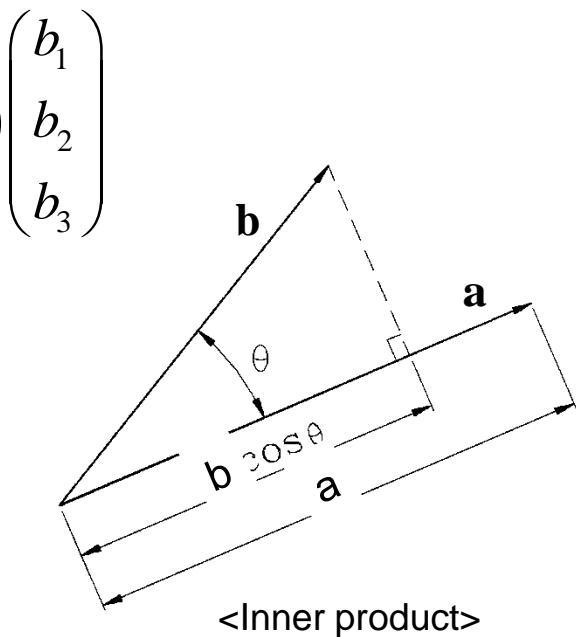
- ◎ Inner product, dot product, scalar product

- $\mathbf{a} \cdot \mathbf{b} \equiv \sum_{i=1}^3 a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a}^T \mathbf{b} = (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

- ◎ Geometric meaning of inner product

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$

- $\mathbf{a} \cdot \mathbf{b} = 0$  means the two vectors are perpendicular.



- ◎ Characteristics of inner product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha (\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \cdot \mathbf{a} \geq 0$$

$$\mathbf{a} \cdot \mathbf{a} = 0 \text{ implies } \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \cdot \mathbf{b} \equiv \sum_{i=1}^3 a_i b_i = \sum_{i=1}^3 b_i a_i = \mathbf{b} \cdot \mathbf{a}$$

- ◎ Norm of vector  $\mathbf{a} : \|\mathbf{a}\|$

- $\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}}$

- $\|\alpha \mathbf{a}\| = |\alpha| \|\mathbf{a}\|$

- $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$

- $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$

- ◎ Miscellaneous

- $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

- $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$



# Algebra of vector – Vector product

- Vector product, cross product

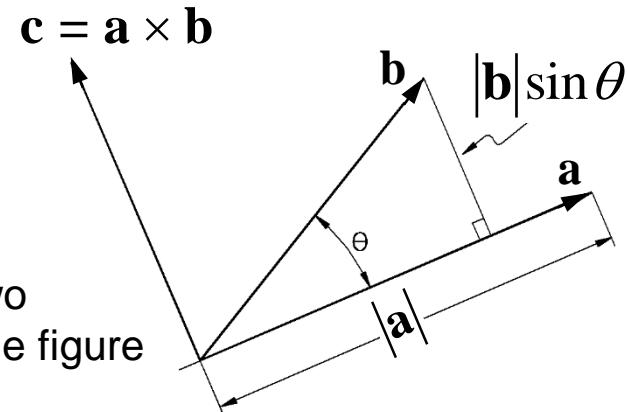
$$\text{O } \mathbf{c} = \mathbf{a} \times \mathbf{b} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$$

- Geometric meaning of vector product

○ Magnitude:  $|\mathbf{c}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$  ( $0 \leq \theta \leq \pi$ )

- Area of parallelogram constructed by the two vectors

○ Direction: Perpendicular to the plane constructed by the two vectors, following the right-hand rule shown in the figure



- Characteristics

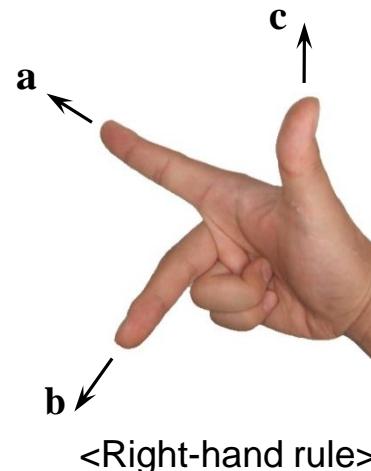
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \rightarrow \mathbf{a} \parallel \mathbf{b}$$



- Miscellaneous

$$\text{O } \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\text{O } \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$



# Euclidean space and linear combination

- ◎  $k$ -dimensional Euclidean space  $\mathbf{R}^k$

- Dimension of vector

= Number of components

- $\mathbf{R}^k$  :  $k$ -dimensional Euclidean vector space,  
or  $k$ -dimensional real number vector space

$$\mathbf{a} = [a_1 \ a_2 \ \cdots \ a_k]^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

$$\mathbf{R}^k \equiv \left\{ \mathbf{a} \mid \mathbf{a} = [a_1, a_2, \dots, a_k]^T; \text{ } a_i \text{'s are real} \right\}$$

$$\mathbf{a} + \mathbf{b} \equiv [a_1 + b_1, a_2 + b_2, \dots, a_k + b_k]^T$$

$$\alpha \mathbf{a} \equiv [\alpha a_1, \alpha a_2, \dots, \alpha a_n]^T$$

$$\mathbf{a} \cdot \mathbf{b} = \sum_{l=1}^k a_l b_l$$

Linearly independent

$$\| \mathbf{a} \| = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}} \quad C_1 [1, 0]^T + C_2 [0, 1]^T = \mathbf{0} \Rightarrow C_1 = C_2 = 0$$

- ◎ Linear combination

$$\sum_{i=1}^n c_i \mathbf{a}_i = c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \cdots + c_n \mathbf{a}_n$$

- ◎ Linearly independent

- The case that the linear combination vanishes only when all  $c_i$ 's are zero.

$$[1, 2, 3]^T$$

$$[1, 0, 1]^T$$

$$[3, 2, 5]^T$$

- ◎ Linearly dependent

- The case that the linear combination vanishes when any  $c_i$  is not zero.

Linearly dependent

$$[1, 2, 3]^T + 2[1, 0, 1]^T - [3, 2, 5]^T = \mathbf{0}$$



# Coordinate system (C.S) and coordinates

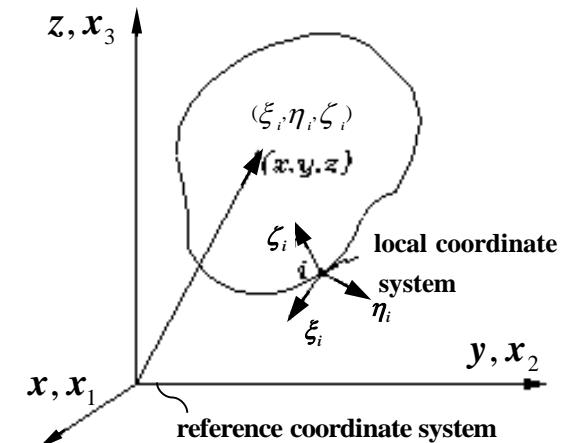
## ○ Coordinates

- Position (components) of a point relative to R.C.S.
- A vector quantity

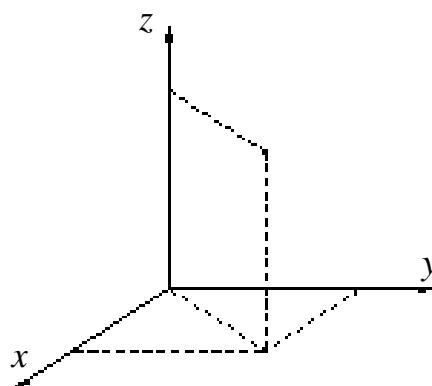
## ○ Reference coordinate system(R.C.S.) and local C.S.

## ○ Orthogonal coordinate system

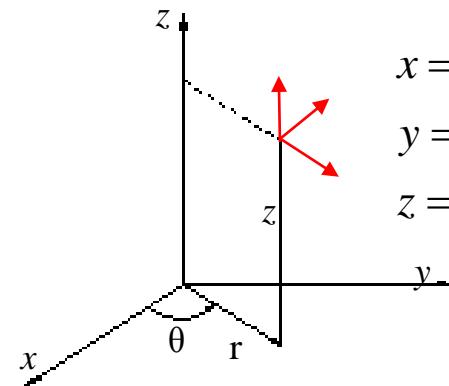
- Rectangular coordinate system
- Cylindrical coordinate system
- Spherical coordinate system



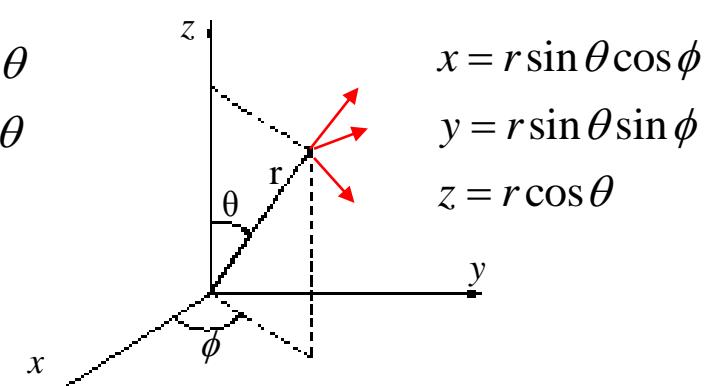
<Coordinate systems and coordinates>



a) R.C.S.



b) C.C.S.



c) S.C.S.

<Typical orthogonal coordinate systems>

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\z &= z\end{aligned}$$

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta\end{aligned}$$



# Examples of vector calculus

◎ Vectors  $\mathbf{a} = [2, -5, 3]^T$  and  $\mathbf{b} = [6, 2, -1]^T$  are given and find the followings:

- a)  $\mathbf{a} + 2\mathbf{b}$       b)  $2\mathbf{a} - \mathbf{b}$       c)  $\mathbf{a} \cdot \mathbf{b}$       d)  $\mathbf{b} \cdot \mathbf{a}$       e) Angle between  $\mathbf{a}$  and  $\mathbf{b}$   
f)  $\mathbf{a} \times \mathbf{b}$       g)  $\mathbf{b} \times \mathbf{a}$       h)  $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$

☞ a)  $\mathbf{a} + 2\mathbf{b} = [14, -1, 1]^T$       b)  $2\mathbf{a} - \mathbf{b} = [-2, -12, 7]^T$   
c)  $\mathbf{a} \cdot \mathbf{b} = 2 \times 6 + (-5) \times 2 + 3 \times (-1) = -1$       d)  $\mathbf{b} \cdot \mathbf{a} = 6 \times 2 + 2 \times (-5) + (-1) \times 3 = -1$

e)  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = \sqrt{2^2 + 5^2 + 3^2} \sqrt{6^2 + 2^2 + 1^2} \cos \theta = \sqrt{1558} \cos \theta$

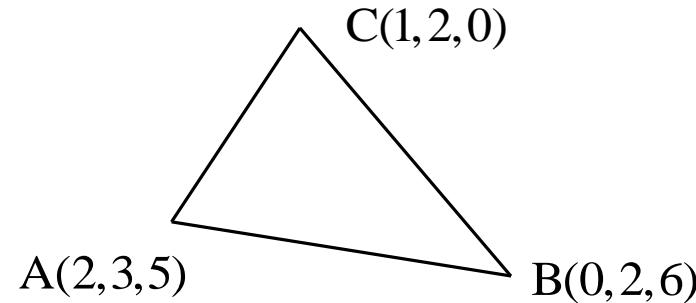
From  $\mathbf{a} \cdot \mathbf{b} = -1$ ,  $\theta = \cos^{-1}(-1/\sqrt{1558}) = 91.45^\circ$ .

- f)  $\mathbf{a} \times \mathbf{b} = (2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \times (6\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 4\mathbf{k} + 2\mathbf{j} + 30\mathbf{k} + 5\mathbf{i} + 18\mathbf{j} - 6\mathbf{i} = -\mathbf{i} + 20\mathbf{j} + 34\mathbf{k}$   
g)  $\mathbf{b} \times \mathbf{a} = (6\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) = -30\mathbf{k} - 18\mathbf{j} - 4\mathbf{k} + 6\mathbf{i} - 2\mathbf{j} - 5\mathbf{i} = \mathbf{i} - 20\mathbf{j} - 34\mathbf{k}$   
h) From a) and b),  $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ .



# Example of vector calculus

◎ Area of triangle

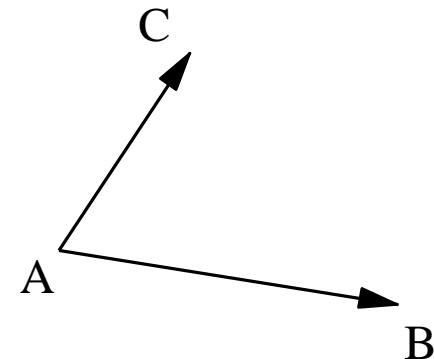


☞  $\overrightarrow{AC} = -\mathbf{i} - \mathbf{j} - 5\mathbf{k}$

$$\overrightarrow{AB} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\Delta ABC = \frac{1}{2} \left| \overrightarrow{AC} \times \overrightarrow{AB} \right|$$

$$\overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -5 \\ -2 & -1 & 1 \end{vmatrix} = (-1 - 5)\mathbf{i} + (10 + 1)\mathbf{j} + (1 - 2)\mathbf{k} = -6\mathbf{i} + 11\mathbf{j} - \mathbf{k}$$

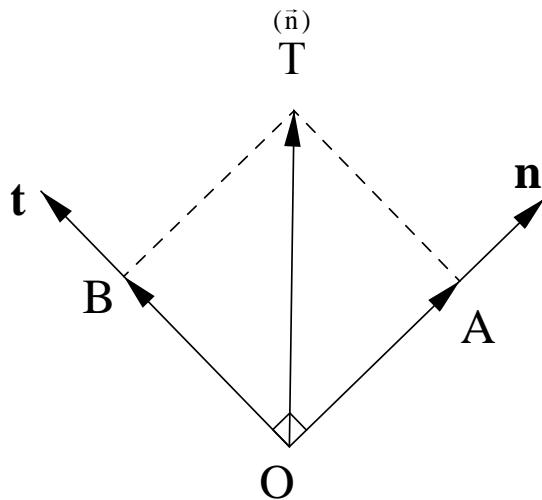


$$\Delta ABC = \frac{1}{2} \times \sqrt{6^2 + 11^2 + 1^2} = 6.285$$



# Example of vector calculus

◎ Calculate vector  $\overrightarrow{OB}$ .



$$\overset{(\vec{n})}{T} = 20\mathbf{i} - 40\mathbf{j}$$

$$\mathbf{n} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\overrightarrow{OB} = ?$$

☞  $\overrightarrow{OA} + \overrightarrow{OB} = \overset{(\vec{n})}{T}$

$$\overrightarrow{OB} = \overset{(\vec{n})}{T} - \overrightarrow{OA}$$

$$\overrightarrow{OA} = |\overrightarrow{OA}| \mathbf{n} = (\overset{(\vec{n})}{T} \cdot \mathbf{n}) \mathbf{n} = (10\sqrt{2} - 20\sqrt{2}) \times \left( \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} \right)$$

$$\overrightarrow{OA} = -10\mathbf{i} - 10\mathbf{j}$$

$$\overrightarrow{OB} = (20\mathbf{i} - 40\mathbf{j}) - (-10\mathbf{i} - 10\mathbf{j}) = 30\mathbf{i} - 30\mathbf{j}$$

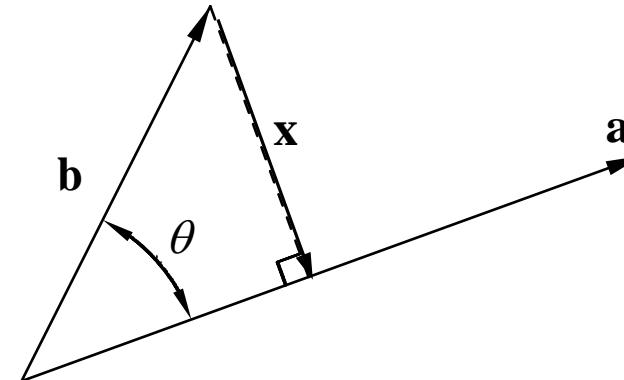


# Example of vector calculus

- For two given vectors  $\mathbf{a}$  and  $\mathbf{b}$ , calculate the vector starting from the end of vector  $\mathbf{b}$  and perpendicularly ending onto the vector  $\mathbf{a}$ .

$$\mathbf{a} = 12\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$



☞ Calculate  $\mathbf{u}_a$ , the unit vector of  $\mathbf{a}$ .

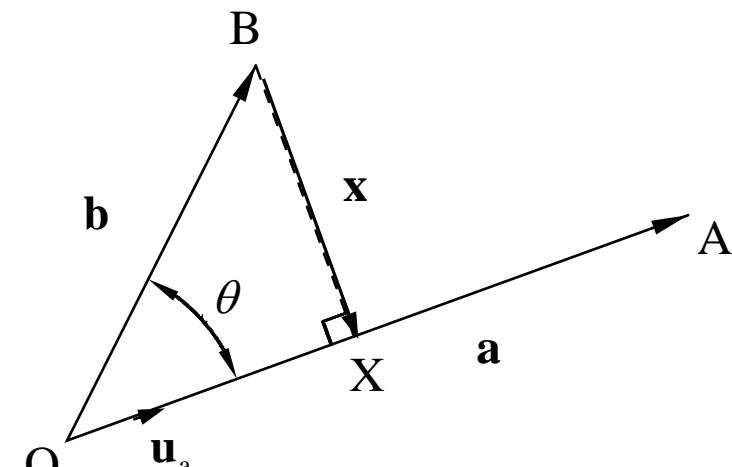
$$\mathbf{u}_a = \frac{12\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}}{\sqrt{12^2 + 3^2 + 4^2}} = \frac{12}{13}\mathbf{i} - \frac{3}{13}\mathbf{j} + \frac{4}{13}\mathbf{k}$$

$$\mathbf{x} = \overrightarrow{BX} = \overrightarrow{OX} - \mathbf{b}$$

$$\overrightarrow{OX} = |\overrightarrow{OX}| \mathbf{u}_a = \overrightarrow{OX} \mathbf{u}_a$$

$$\overrightarrow{OX} = \mathbf{u}_a \cdot \mathbf{b} = |\mathbf{b}| \cos \theta = -\frac{24}{13} - \frac{9}{13} + \frac{20}{13} = -1$$

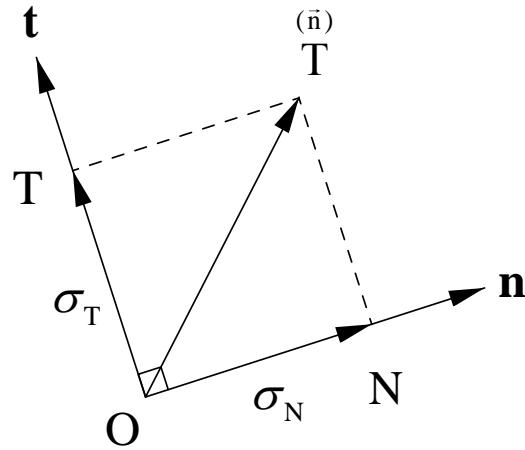
$$\mathbf{x} = -1 \times \left( \frac{12}{13}\mathbf{i} - \frac{3}{13}\mathbf{j} + \frac{4}{13}\mathbf{k} \right) - (-2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) = \frac{14}{13}\mathbf{i} - \frac{36}{13}\mathbf{j} - \frac{69}{13}\mathbf{k}$$





# Example of vector calculus

◎ Calculate  $\sigma_T$  in the three-dimensional space.



$$T = 27\mathbf{i} - 18\mathbf{j} + 36\mathbf{k}$$

$$\mathbf{n} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}$$

$$\sigma_T = ?$$

☞  $\overrightarrow{ON} + \overrightarrow{OT} = \overset{(n)}{\overrightarrow{T}}$

$$\sigma_T = |\overrightarrow{OT}| = \left| \overset{(n)}{\overrightarrow{T}} - \overrightarrow{ON} \right|$$

$$\overrightarrow{ON} = |\overrightarrow{ON}| \mathbf{n} = (\overset{(n)}{\overrightarrow{T}} \cdot \mathbf{n}) \mathbf{n} = (9\sqrt{3} - 6\sqrt{3} + 12\sqrt{3}) \times \left( \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} \right) = 15\mathbf{i} + 15\mathbf{j} + 15\mathbf{k}$$

$$\sigma_T = |(27\mathbf{i} - 18\mathbf{j} + 36\mathbf{k}) - (15\mathbf{i} + 15\mathbf{j} + 15\mathbf{k})| = |12\mathbf{i} - 33\mathbf{j} + 21\mathbf{k}|$$

$$\sigma_T = 40.915$$

## **2.2 Matrix and Linear Algebra**



# Definition of matrix and terminology

- Matrix: Rectangular array of numbers, called elements

- $m \times n$  matrix :  $\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- Upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- Terminologies :

- Row vector :  $1 \times n$  matrix

- Column vector :  $m \times 1$  matrix

- Square matrix :  $n \times n$  matrix

- Off-diagonal term :  $a_{ij}$  ( $i \neq j$ )

- Diagonal term :  $a_{ii}$  ( $i = 1, 2, \dots, n$ ) for an  $n \times n$  square matrix

- Zero matrix

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- Diagonal matrix

$$\mathbf{D} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- Lower triangular matrix

$$\mathbf{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- Unit (identity) matrix

$$\mathbf{I} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

↑  
Kronecker delta



# Definition of terminologies continued

- **Submatrix:** A matrix made by deleting some rows or columns from the original matrix
- **Principal submatrix:** A submatrix made by deleting simultaneously some  $i$ -th row(s) and the same  $i$ -th column(s) of a square matrix
- **Transpose of a matrix  $\mathbf{A}$ ,**  $\mathbf{A}^T$  : A matrix of which  $(i,j)$  component is equal to the  $(j,i)$  component of matrix  $\mathbf{A}$ ,  $a_{ij}^T = a_{ji}$
- **Symmetric matrix:**  $\mathbf{A}^T = \mathbf{A}$ ,  $a_{ij} = a_{ji}$
- **Skew-symmetric matrix:**  $\mathbf{A}^T = -\mathbf{A}$ ,  $a_{ij} = -a_{ji}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- **Rank:** Number of independent rows = number of independent columns
- **Singular matrix:**  $n$  square matrix of which rank is less than  $n$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$\text{Rank}(A) = 2$$

Principal submatrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \begin{bmatrix} a_{22} \\ a_{32} \end{bmatrix} \begin{bmatrix} a_{33} \end{bmatrix}$$

Skew-symmetric

$$\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ a_{12} & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$



# Matrix and Vector

---

◎ Expression of  $m \times n$  matrix using  $m$  row vectors or  $n$  column vectors

$$\mathbf{A} = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n) \quad \mathbf{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} \quad \mathbf{b}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$



# Addition of matrixes and multiplication of real number and matrix

## ◎ Definition of matrixes

- $\mathbf{C} \equiv \mathbf{A} + \mathbf{B} \rightarrow c_{ij} \equiv a_{ij} + b_{ij}$

## ◎ Multiplication of real number and matrix

- $\mathbf{C} \equiv \alpha \mathbf{A} \rightarrow c_{ij} \equiv \alpha a_{ij}$

## ◎ Properties

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \qquad \qquad a_{ij} + b_{ij} \equiv b_{ij} + a_{ij} \Leftrightarrow \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- $\mathbf{A} + \mathbf{0} = \mathbf{A} \qquad \qquad \circ \mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$
- $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
- $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A} \qquad \circ 1\mathbf{A} = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \quad \circ (\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$



# Product of matrixes

## ◎ Definition of product of matrixes $\mathbf{A}$ and $\mathbf{B}$

- $\mathbf{C} = \mathbf{A} \mathbf{B}$ : Product of  $m \times p$  matrix  $\mathbf{A} = [a_{ij}]$  and  $q \times n$  matrix  $\mathbf{B} = [b_{ij}]$

- $p = q \equiv l$  is the essential requirement for  $c_{ij}$  to be defined.
- $c_{ij} = \sum_{k=1}^l a_{ik} b_{kj} = \mathbf{a}_i \cdot \mathbf{b}_j = \mathbf{a}_i^T \mathbf{b}_j$

## ◎ Properties of matrix product

- $(\alpha \mathbf{A})\mathbf{B} = \alpha(\mathbf{A}\mathbf{B}) = \mathbf{A}(\alpha\mathbf{B})$

- $\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$

- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$

- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

- $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

$$i \begin{pmatrix} & & j \\ & \vdots & \\ \dots & c_{ij} & \dots \\ & \vdots & \end{pmatrix} = i \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots a_{il} \\ \vdots & \vdots & \vdots & \vdots \\ & & & \end{pmatrix} \begin{pmatrix} \cdots b_{1j} \cdots \\ \cdots b_{2j} \cdots \\ \cdots b_{3j} \cdots \\ \vdots \\ \cdots b_{lj} \cdots \end{pmatrix}$$

$m \times n$  matrix       $m \times l$  matrix       $l \times n$  matrix

- In general,  $\mathbf{AB} \neq \mathbf{BA}$  and  $\mathbf{AB} = \mathbf{0}$  does not always mean  $\mathbf{A} = \mathbf{0}$  or  $\mathbf{B} = \mathbf{0}$

- $\mathbf{IA} = \mathbf{A}$ ,  $\mathbf{AI} = \mathbf{A}$ ,  $\mathbf{Ix} = \mathbf{x}$

## ◎ Example: Show $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ using the following two matrixes

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -5 & 3 \\ 4 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & -2 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$$

☞  $\mathbf{AB} = \begin{bmatrix} 1 & -8 \\ 0 & 7 \\ 22 & -3 \end{bmatrix}$  gives  $(\mathbf{AB})^T = \begin{bmatrix} 1 & 0 & 22 \\ -8 & 7 & -3 \end{bmatrix}$ ,

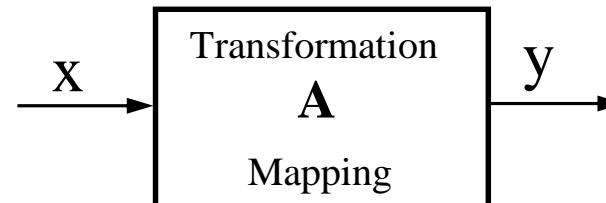
$$\mathbf{B}^T \mathbf{A}^T = \begin{bmatrix} 5 & 2 & 0 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -2 & -5 & 1 \\ -4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 22 \\ -8 & 7 & -3 \end{bmatrix}. \text{ Therefore } (\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T .$$



# Role of matrix

- ◎ Role of matrix :  $\mathbf{Ax} = \mathbf{y}$

- Mathematical operator
- Transfer function or transformation



- ◎ Law of coordinate transformation of a vector :

$$\textcircled{O} \quad \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} F_{x'} \cos\theta &= F_x + F_y \sin\theta \\ F_{y'} \cos\theta &= F_y - F_x \sin\theta \end{aligned} \rightarrow \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix}$$

- Transformation  $\mathbf{T} = [\mathbf{t}_1, \mathbf{t}_2]$     $\mathbf{t}_1 = [\cos\theta, -\sin\theta]^T$     $\mathbf{t}_2 = [\sin\theta, \cos\theta]^T$   
matrix

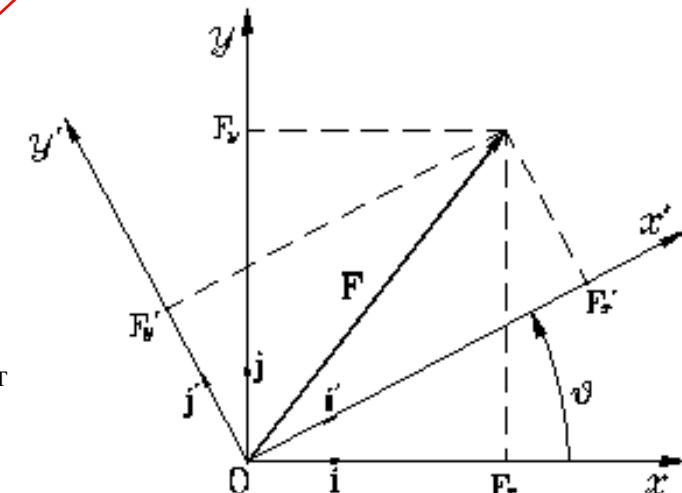
$$|\mathbf{t}_1| = |\mathbf{t}_2| = 1, \quad \mathbf{t}_1 \cdot \mathbf{t}_2 = 0 \leftarrow \text{Orthonormal matrix}$$

$$\mathbf{T}^{-1} = \mathbf{T}^T$$

Inverse matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{|\mathbf{T}|} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \mathbf{T}^T$$

Transformation  
matrix



<Coordinate transformation>

$$\textcircled{O} \quad \begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix}$$



# Application of the role of matrix

## ○ Matrix in mechanics

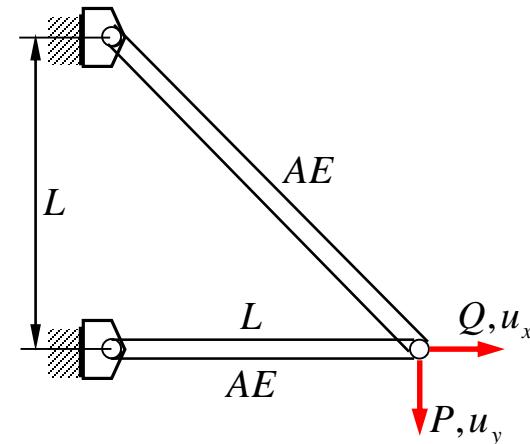
### ○ Displacement-load relation

$$\frac{L}{AE} \begin{bmatrix} 1 & -1 \\ -1 & 2\sqrt{2}+1 \end{bmatrix} \begin{bmatrix} \frac{Q}{P} \\ \frac{u_x}{u_y} \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\mathbf{AF}=\mathbf{U}$$

$$\frac{AE}{2\sqrt{2}L} \begin{bmatrix} 2\sqrt{2}+1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{u_x}{u_y} \\ \frac{Q}{P} \end{bmatrix} = \begin{bmatrix} Q \\ P \end{bmatrix}$$

$$\mathbf{KU}=\mathbf{F}$$



### ○ Displacement-load relation

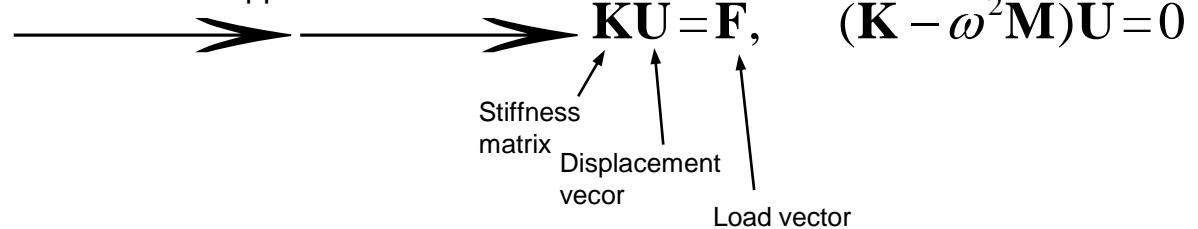
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0$$

Approximate approach to differential equation

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0$$

+ Finite element approximation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

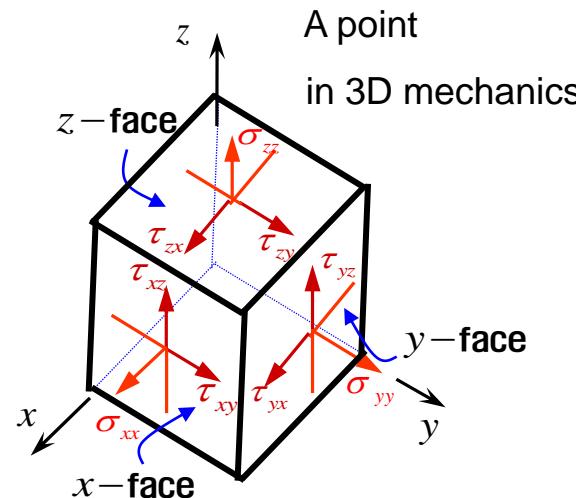




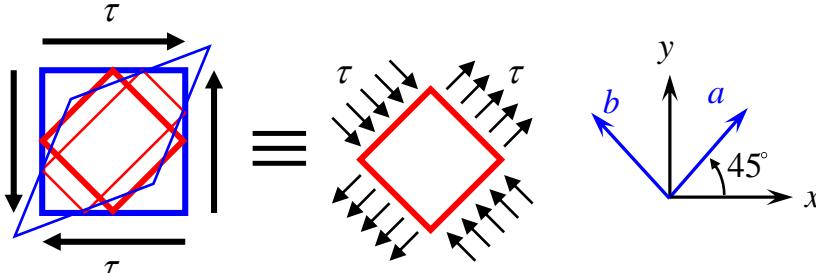
# Expression of the second order of tensor

## ◎ Stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$



## ◎ Coordinate transformation

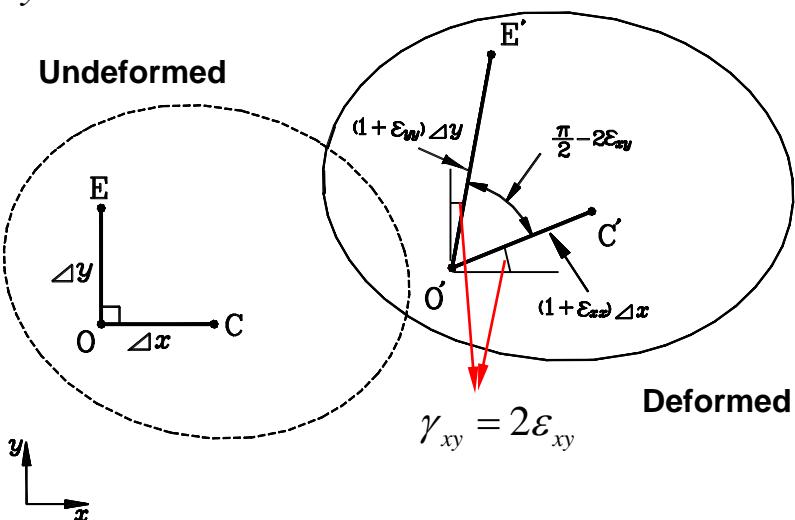


$$\begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \begin{pmatrix} \varepsilon_{x'} & \varepsilon_{x'y'} \\ \varepsilon_{y'x'} & \varepsilon_{y'} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

## ◎ Strain tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$





# Determinant of a matrix

- Determinant of a  $2 \times 2$  matrix:

$$\begin{aligned} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \mathbf{Ax} = \mathbf{b} \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ & = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{pmatrix} \end{aligned}$$

- Determinant of a  $3 \times 3$  matrix :

$$O D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If  $\det \mathbf{A} = 0.0$ ,  
 $\mathbf{A}$  is singular.

- Determinant of an  $n \times n$  matrix :

$$O D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots = a_{11}M_{11} - a_{12}M_{12} \cdots$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots$$

$$O D = \sum_{i=1}^n a_{ji} C_{ji} = \sum_{i=1}^n a_{ij} C_{ij} \quad (j = 1, 2, \dots, n), \quad C_{ij} = (-1)^{i+j} M_{ij} \quad (C_{ij} : \text{cofactor})$$

$$O D = \sum_{i=1}^n (-1)^{i+j} a_{ji} M_{ji} = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij}$$

O  $M_{ij}$ : Minor, Determinant of the  $(n-1) \times (n-1)$  submatrix,  $i$ -row and  $j$ -column were removed.



# Characteristics of determinant of matrix

## ◎ Characteristics

- ①  $|\mathbf{A}| = |\mathbf{A}^T|$
- ②  $|\mathbf{AB}| = |\mathbf{BA}| = |\mathbf{A}| |\mathbf{B}|$
- ③ If a row or a column of a matrix is multiplied by  $c$ ,  
determinant of the newly formed matrix is  $c$ -multiple of the original value.
- ④ When two rows (or columns) are changed, the resulting determinant becomes negative of the original value.
- ⑤ When a row (or column) is added by any other row (or column) multiplied by a constant, the resulting determinant does not change.
- ⑥ When row vectors (or column vectors) of a matrix are linearly dependent, its determinant vanishes.

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

## ◎ Applications

$$\textcircled{1} \quad D' = \det \mathbf{A}' = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \det \mathbf{A}$$

$$\textcircled{2} \quad D'' = \det \mathbf{A}'' = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = - \det \mathbf{A}$$

$$\textcircled{3} \quad \tilde{D} = \det \tilde{\mathbf{A}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{11} + a_{21} & 2a_{12} + a_{22} & 2a_{13} + a_{23} \end{vmatrix} = 0$$



# Quadratic form and kind of matrices

◎ Quadratic form:  $f = \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x}$

$$U = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u}, \quad K = \frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v}$$

◎ Positive definiteness of  $\mathbf{A}$

① When all eigenvalues are positive.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{22} \\ a_{33} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{22} \\ a_{33} \end{bmatrix}$$

② When determinants of all principal submatrices are all positive.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{33} \end{bmatrix}$$

③  $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$  for any  $\mathbf{x}$  and  $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$  only when  $\mathbf{x} = \mathbf{0}$ .

◎ Positive semi-definiteness of  $\mathbf{A}$

Positive definite

① When all eigenvalues are non-negative.

$$(x_1 - x_2)^2 + x_2^2 = (x_1, x_2) \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

② When determinants of all principal submatrices are all non-negative.

③  $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$  for any  $\mathbf{x}$ .

◎ Positive definiteness of  $\mathbf{A}$  = Negative definiteness of  $-\mathbf{A}$

Positive semi-definite

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 = (x_1, x_2, x_3) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$$



# Inverse matrix

- ⊕ Inverse matrix of  $n \times n$  matrix  $\mathbf{A}$ :  $\mathbf{A}^{-1}$

- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$     or     $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

- $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [\mathbf{C}_{ij}]^T = \frac{1}{\det \mathbf{A}} [\mathbf{A}_{ij}]^*$

$[\mathbf{A}_{ij}]^* = [\mathbf{C}_{ij}]^T$  : Adjoint of matrix  $\mathbf{A}$

$$[\mathbf{A}_{ij}]^* = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21} & \cdots & \mathbf{C}_{n1} \\ \mathbf{C}_{12} & \mathbf{C}_{22} & \cdots & \mathbf{C}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{1n} & \mathbf{C}_{2n} & \cdots & \mathbf{C}_{nn} \end{bmatrix}$$

- ⊕ Orthonormal matrix and transformation matrix

- If  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{D}$ , i.e., diagonal matrix, the matrix  $\mathbf{A}$  is **orthogonal matrix**.

- If  $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ ,  $\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij}$ , the matrix  $\mathbf{A}$  is **orthonormal matrix**, i.e.,  $\mathbf{A}^{-1} = \mathbf{A}^T$ .

- Transformation matrix is orthonormal, i.e.,  $\mathbf{T}\mathbf{T}^T = \mathbf{I}$ . Therefore,  $\mathbf{T}^{-1} = \mathbf{T}^T$ .

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{-1} = \frac{1}{|\mathbf{T}|} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \mathbf{T}^T$$

- ⊕ Inverse of matrix  $\mathbf{A}$  multiplied by  $\mathbf{B}$

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

- $(\mathbf{ABCD} \cdots)^{-1} = \cdots \mathbf{D}^{-1}\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

- ⊕ Linear equation

- $\mathbf{Ax} = \mathbf{b}$

- $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$ ,  $\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$ ,  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

- ※ **Kronecker delta**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\mathbf{I} = \begin{bmatrix} \delta_{ij} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0.85 & -0.53 \\ 0.53 & 0.85 \end{pmatrix} \begin{pmatrix} 0.85 & 0.53 \\ -0.53 & 0.85 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



# Similarity transformation

- What is similarity transformation?

$$\tilde{\mathbf{A}} = \mathbf{R} \mathbf{A} \mathbf{R}^{-1}$$

$$\begin{pmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

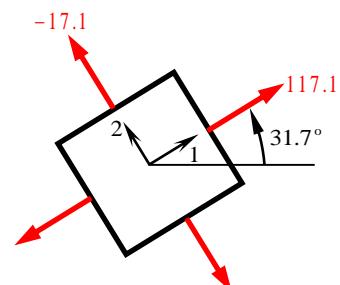
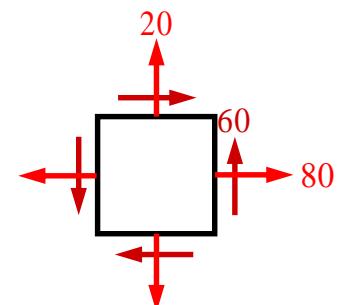
- Characteristics of the similarity transformation

- Eigenvalues of  $\tilde{\mathbf{A}}$  and  $\mathbf{A}$  are identical.
- Relationship of eigenvectors :  $\tilde{\mathbf{x}} = \mathbf{R}^{-1} \mathbf{x}$   
( $\mathbf{x}$ : eigenvector of matrix  $\mathbf{A}$ ,  $\tilde{\mathbf{x}}$ : eigenvector of matrix  $\tilde{\mathbf{A}}$ )

- Application of transformation matrix

$$\begin{aligned} \sigma_{i'j'} &= T_{i'p} T_{j'q} \sigma_{pq} \\ &= T_{i'p} \sigma_{pq} T_{j'q} \end{aligned} \rightarrow \begin{bmatrix} \sigma_{i'p'} \end{bmatrix} = \begin{bmatrix} T_{i'p} \end{bmatrix} \begin{bmatrix} \sigma_{pq} \end{bmatrix} \begin{bmatrix} T_{j'q} \end{bmatrix}$$

$$[\tilde{\sigma}] = [T][\sigma][T]^T = [T][\sigma][T]^{-1}$$



$$\begin{bmatrix} 0.85 & -0.53 \\ 0.53 & 0.85 \end{bmatrix} \begin{bmatrix} 80 & 60 \\ 60 & 20 \end{bmatrix} \begin{bmatrix} 0.85 & 0.53 \\ -0.53 & 0.85 \end{bmatrix} = \begin{bmatrix} 117.1 & 0 \\ 0 & -17.1 \end{bmatrix}$$



# Eigenvalue problem

## ◎ Homogeneous linear equation: $\mathbf{Ax} = \mathbf{0}$

- $\mathbf{x} = \mathbf{0}$ : Trivial solution, meaningless solution
- IF  $|\mathbf{A}| = 0$ , there is  $\mathbf{x} \neq 0$  but  $\mathbf{Ax} = \mathbf{0}$ .

$$\begin{aligned}x - 2y &= 0 \\x - cy &= 0\end{aligned} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ 1 & -c \end{vmatrix} = 0 \rightarrow c = 2 \Rightarrow x:y = \frac{2\sqrt{5}}{5} : \frac{\sqrt{5}}{5}$$

## ◎ Eigenvalue problem : $\mathbf{Ax} = \lambda \mathbf{x}$ or $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$

- Rows or columns of the  $n \times n$  matrix  $(\mathbf{A} - \lambda \mathbf{I})$  should be linearly dependent.
- $\lambda$ : Eigenvalue or characteristic value
- $\mathbf{x}$ : Eigenvector or characteristic vector

## ◎ Characteristic equation: $|\mathbf{A} - \lambda \mathbf{I}| = 0$

- Requirement that rank of  $\mathbf{A} - \lambda \mathbf{I}$  is less than  $n$ , or that  $\mathbf{A} - \lambda \mathbf{I}$  is singular.
- Non-linear equation of order  $n$ .
- If the matrix  $\mathbf{A}$  is symmetric, the  $n$  real-value solutions exist, i.e.,  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ .

## ◎ Orthogonality of eigenvectors : $\mathbf{x}_i \cdot \mathbf{x}_j = 0$

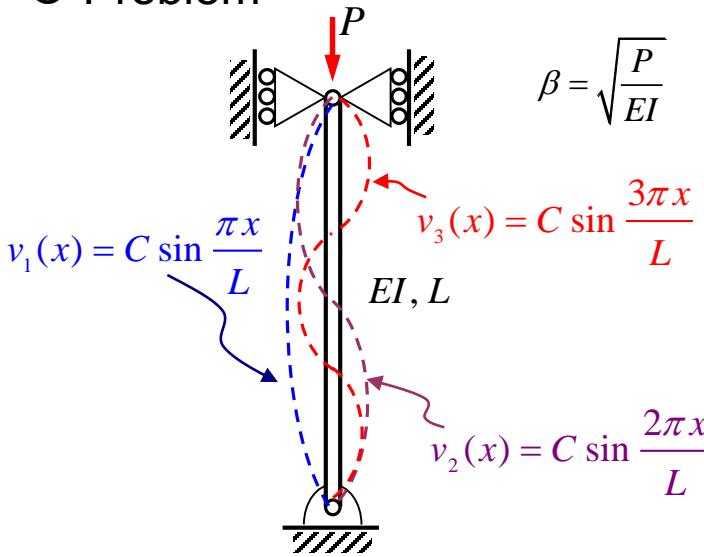
$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}$$

- $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_i \rightarrow \mathbf{x}_j^T \mathbf{A} \mathbf{x}_i = \lambda_i \mathbf{x}_j^T \mathbf{x}_i, (\mathbf{x}_i^T \mathbf{A} \mathbf{x}_j) = \lambda_j \mathbf{x}_i^T \mathbf{x}_j$
- $\mathbf{x}_j^T (\mathbf{A} - \mathbf{A}^T) \mathbf{x}_i = (\lambda_i - \lambda_j) \mathbf{x}_i \cdot \mathbf{x}_j$
- If  $\mathbf{A} - \mathbf{A}^T = \mathbf{0}, (\lambda_i - \lambda_j) \mathbf{x}_i \cdot \mathbf{x}_j = 0 \rightarrow \mathbf{x}_i \cdot \mathbf{x}_j = 0$



# Eigenvalue problem – buckling

- Problem



- GE and homogeneous solution

$$v^{(4)}(x) + \frac{P}{EI} v''(x) = 0, \quad \beta^2 = \frac{P}{EI}$$

$$v(x) = C_1 + C_2 x + C_3 \sin \beta x + C_4 \cos \beta x$$

- BCs

$$v(0) = 0, \quad v(L) = 0$$

$$M_b(0) = 0 \rightarrow v''(0) = 0$$

$$M_b(L) = 0 \rightarrow v''(L) = 0$$

- Solving

$$v''(x) = -C_3 \beta^2 \sin \beta x - C_4 \beta^2 \cos \beta x$$

$$v(0) = C_1 + C_4 = 0$$

$$v(L) = C_1 + C_2 L + C_3 \sin \beta L + C_4 \cos \beta L = 0$$

$$v''(0) = -C_4 \beta^2 = 0$$

$$v''(L) = -C_3 \beta^2 \sin \beta L - C_4 \beta^2 \cos \beta L = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & L & \sin \beta L & \cos \beta L \\ 0 & 0 & 0 & \beta^2 \\ 0 & 0 & \beta^2 \sin \beta L & \beta^2 \cos \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sin \beta L = 0 \rightarrow \beta L = n\pi \rightarrow \beta = n\pi / L \quad (n=1, 2, \dots, \infty)$$

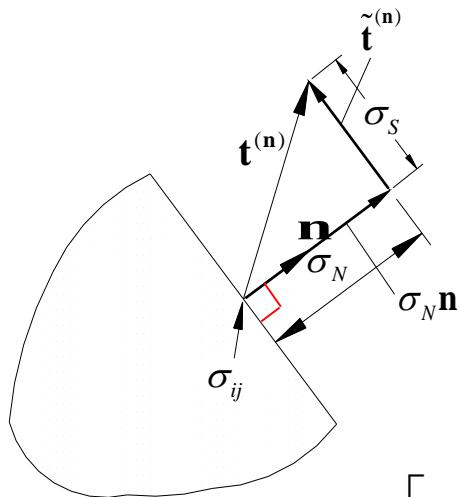
$$v(x) = C_3 \sin \beta x$$

$$\sqrt{\frac{P_n}{EI}} L = n\pi \rightarrow P_n = n^2 \pi^2 \frac{EI}{L^2} \rightarrow v_n(x) = C \sin \frac{n\pi x}{L}$$

$$\sqrt{\frac{P_{cr}}{EI}} L = \pi \rightarrow P_{cr} = \pi^2 \frac{EI}{L^2} \quad \text{← Buckling load}$$



# Eigenvalue problem – Principal stresses



$$\begin{bmatrix} t_x^{(n)} \\ t_y^{(n)} \\ t_z^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Cauchy's formula

$$\boldsymbol{\sigma}' = \mathbf{T} \boldsymbol{\sigma} \mathbf{T}^{-1}$$

Coordinate transformation rule  
of tensor of order 2  
-Similarity transformation-

○ Problem:

$$\begin{bmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \sigma \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Characteristic equation

○ Solution:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma \end{vmatrix} = -[\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3] = 0 \rightarrow \sigma = \sigma_1, \sigma_2, \sigma_3$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \sigma_i \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \Rightarrow \mathbf{n} = \mathbf{n}^{(i)}$$

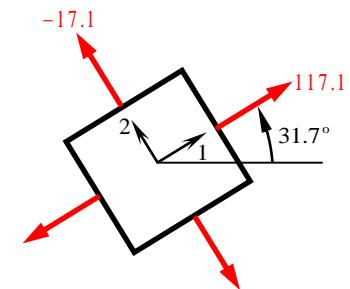
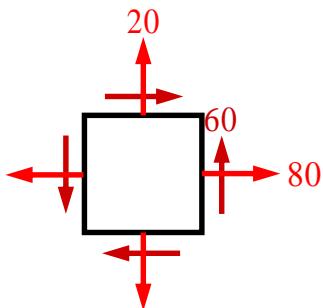
Invariants (stress, strain, etc.)

First, second, third invariant

Principal direction

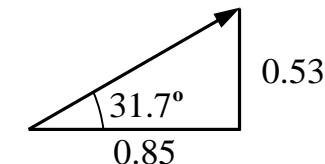
Principal value

# Example of determining principal values of stress



- Eigenvalue problem and its characteristic equation, eigenvalue (principal stress)

$$\begin{bmatrix} 80 & 60 \\ 60 & 20 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \rightarrow \begin{vmatrix} 80 - \sigma & 60 \\ 60 & 20 - \sigma \end{vmatrix} = 0 \rightarrow \begin{aligned} \sigma_1 &= 117.1 \\ \sigma_2 &= -17.1 \end{aligned}$$



- Eigenvector (Principal direction)

i)  $\sigma_1 = 117.1$

$$\begin{bmatrix} 80 - 117.1 & 60 \\ 60 & 20 - 117.1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -37.1n_1 + 60n_2 = 0$$

ii)  $\sigma_1 = -17.1$

$$\begin{bmatrix} 80 + 17.1 & 60 \\ 60 & 20 + 17.1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 60n_1 + 37.1n_2 = 0$$

$$\theta = \tan^{-1} \frac{0.53}{0.85} = 31.7^\circ$$

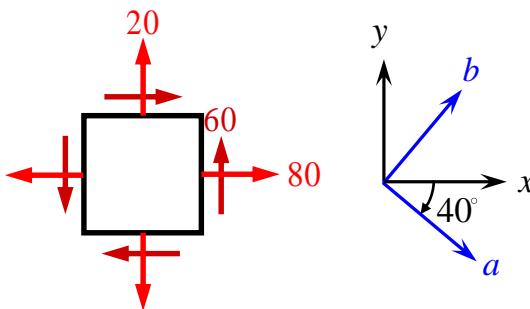
$$\mathbf{n}^{(1)} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0.85 \\ 0.53 \end{pmatrix}$$

$$\mathbf{n}^{(2)} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^{(2)} = \begin{pmatrix} -0.53 \\ 0.85 \end{pmatrix}$$



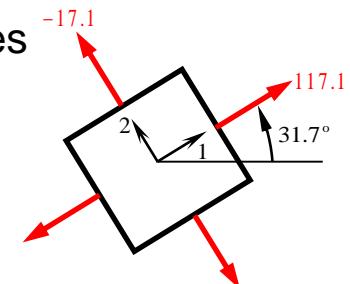
# Application of Mohr's circle

## Given stress state



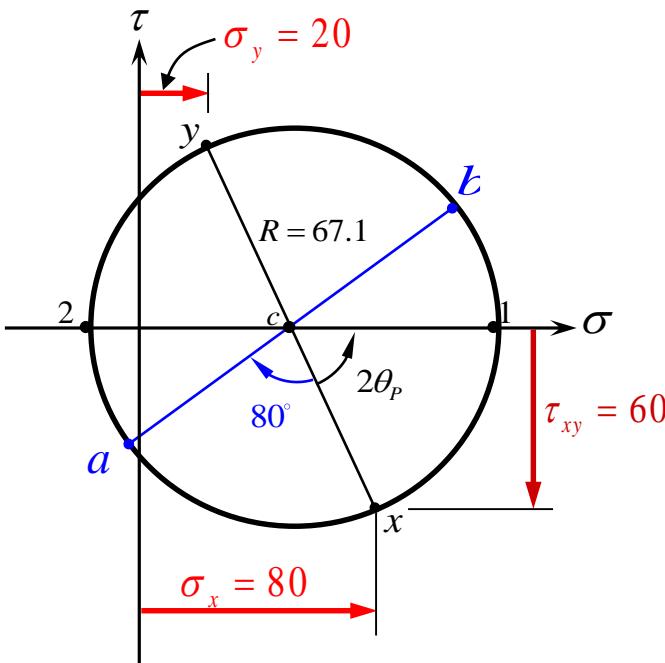
## Calculation of principal stresses

- $\sigma_1 = c + R = 117.1$
- $\sigma_2 = c - R = -17.1$

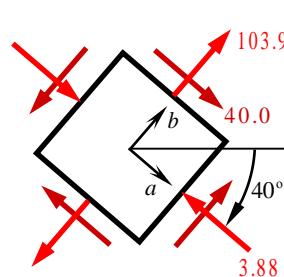


Principal stresses and directions

## Mohr's circle



## $\sigma_a, \sigma_b, \tau_{ab}$ ?



- $R = \sqrt{\left(\frac{80-20}{2}\right)^2 + 60^2} = 61.7$
- $2\theta_P = \sin^{-1} \frac{\tau_{xy}}{R} = 63.43^\circ \Rightarrow \theta_P = 31.7^\circ$
- $\sigma_a = c - R \cos(180^\circ - 80^\circ - 2\theta_P)$   
 $= \frac{80+20}{2} - 67.1 \cos 36.57^\circ = -3.88$
- $\sigma_b = 50 + 67.1 \cos 36.57^\circ = 103.9$
- $\tau_{ab} = \sin 36.57^\circ = 40.0$

## **2.3 Function and Differentiation**



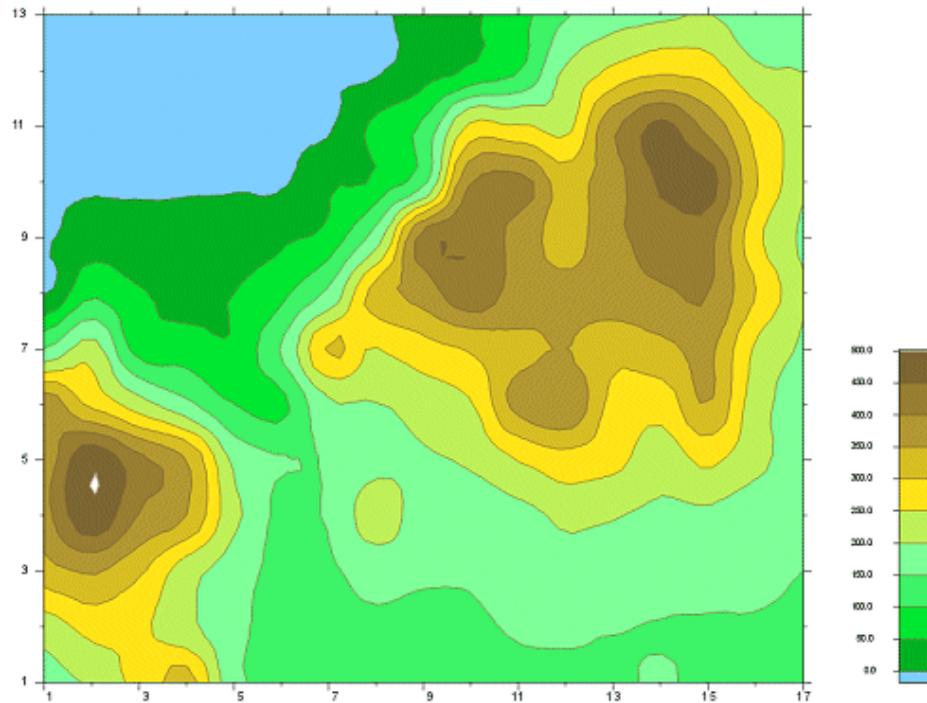
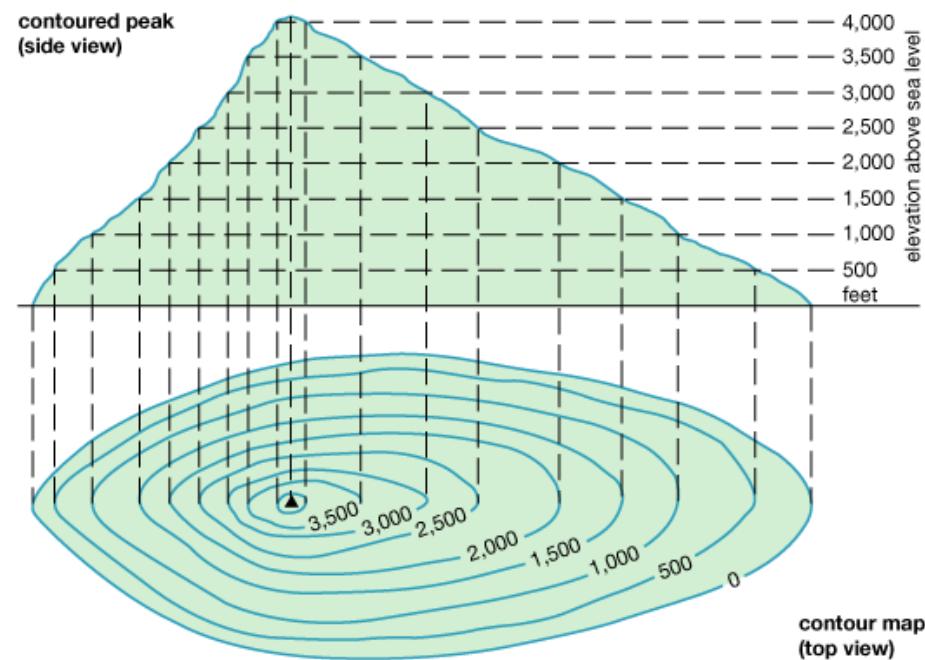
# Quantification of a function in 2D

Height of mountain

Atmospheric pressure

$$h = h(x, y)$$

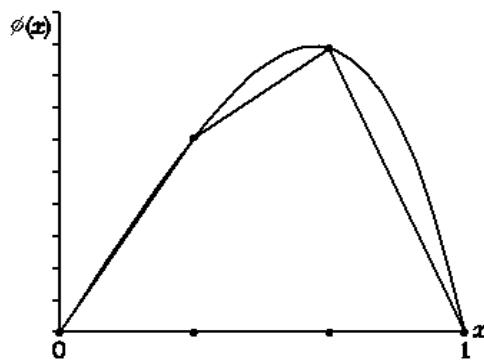
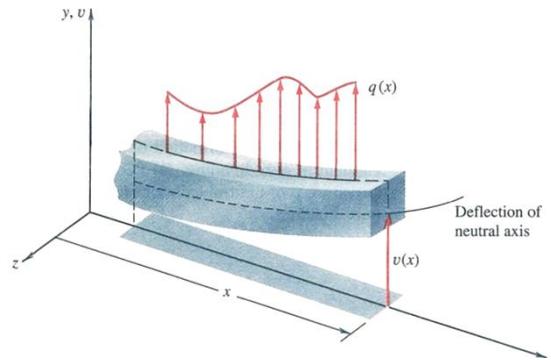
$$p = p(x, y)$$



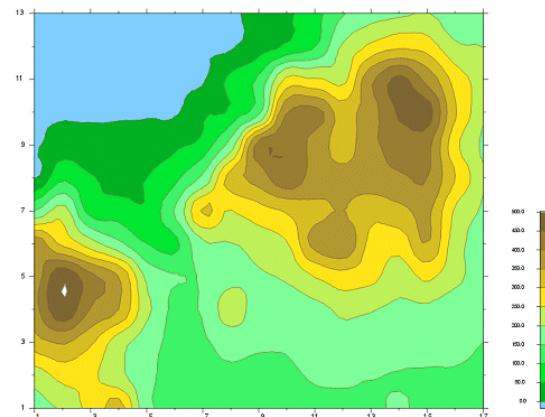


# Mechanics and unknown functions

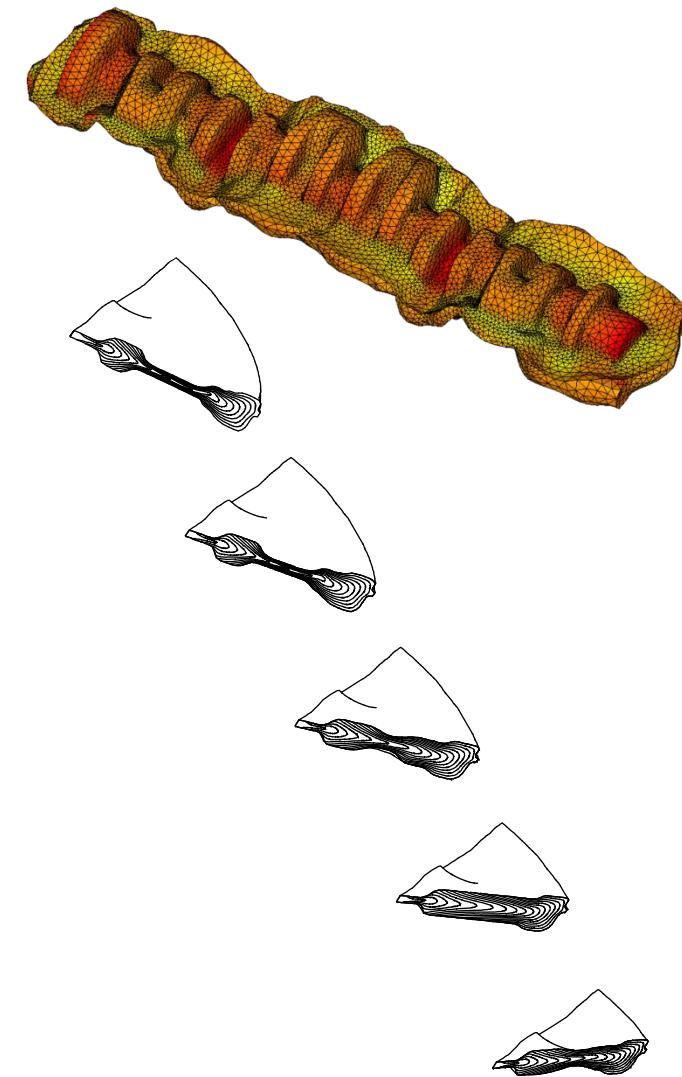
## ○ 1-Dimensional



## ○ 2-Dimensional



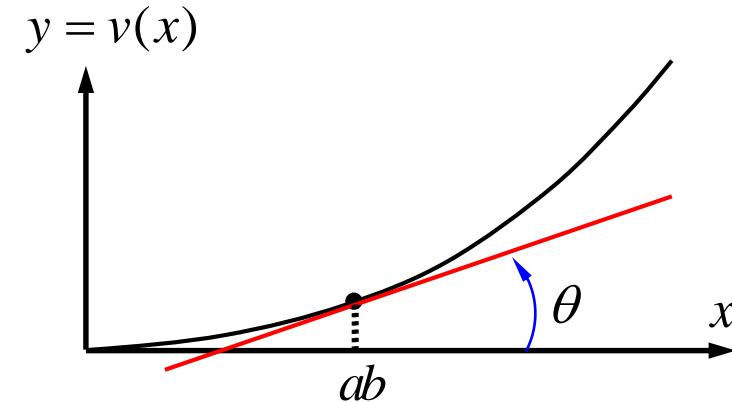
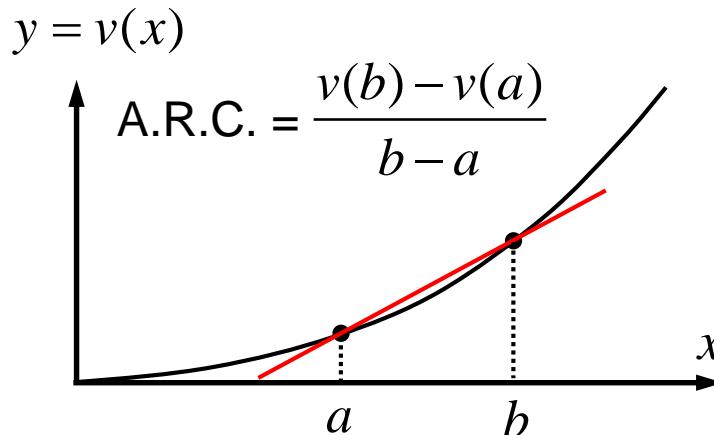
## ○ 3-Dimensional





# Rate of change

## ○ Average rate of change



## ○ Derivative

$$v'(x) = \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} = \frac{dv}{dx}$$

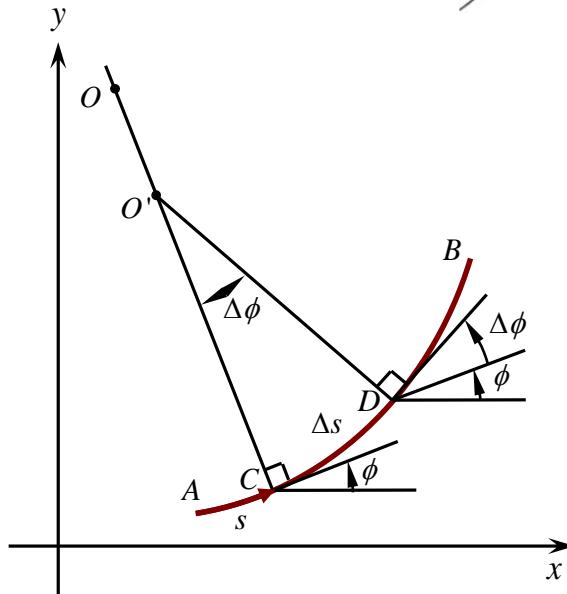
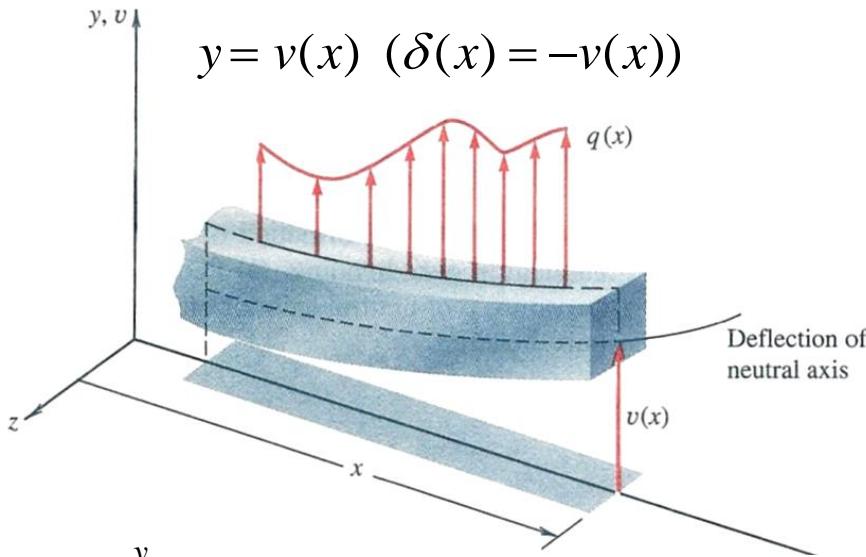
$$\begin{aligned} v'(a) &= \lim_{b \rightarrow a} \frac{v(b) - v(a)}{b - a} \\ &= \lim_{\Delta x \rightarrow 0} \frac{v(a + \Delta x) - v(a)}{\Delta x} \\ &= \tan \theta \end{aligned}$$

- Mexicans in Mexico city do not recognize its height. Height itself does not make things slip while the slope decides the slipping speed of them.
- Dust wind in Africa does not affect on me. However, that in Gobi desert has much influence on me, implying that rate of change is of great importance.



# Ordinary differentiation and slope

## Beam deflection



## Differentiation

$$v'(x) = \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} = \frac{dv}{dx}$$

## $v(x)$ in beam theory

$$EIv''(x) = M_b(x) \quad M_b'(x) = -V(x)$$

$$V(x) = -(EIv''(x))'$$

$$\kappa = \frac{1}{\rho} = \frac{d\phi}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\Delta\phi}{\Delta s}$$

$$\kappa = \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot \frac{dx}{ds} = \frac{v''(x)}{(1 + v'^2(x))^{\frac{3}{2}}} \approx v''(x)$$



# Ordinary differentiation

◎ Average rate of change :  $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$

◎ Ordinary differentiation

- Setting  $\Delta x \rightarrow 0$  on the average rate of change

- Only for functions with one dependent variable

\* Partial differentiation: For functions with more than 2 independent variables

- Ordinary differentiation  $y = f(x) \quad y' = \frac{dy}{dx}$

of

- $y' = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) = \frac{df}{dx}$

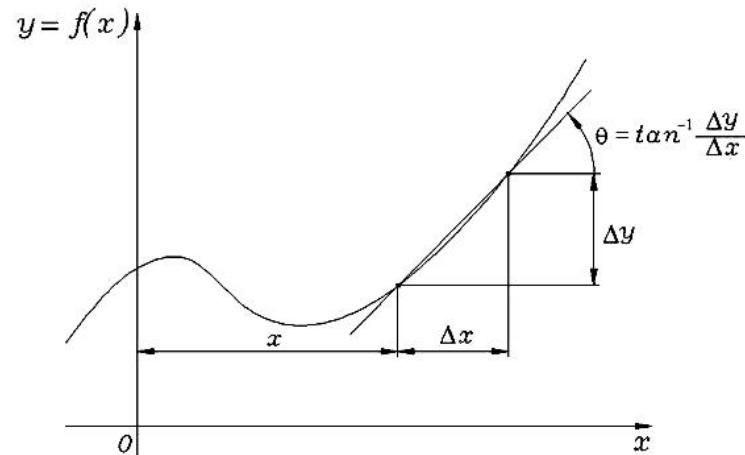
- It is defined when  $\lim_{\Delta x \rightarrow 0+} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0-} \frac{\Delta y}{\Delta x}$

- Geometric meaning: Slope at a point on curve

- Mathematical meaning: Instant rate of change

- Name: First derivative

- Dimension of  $y'$        $[y'] : [y] / [x]$        $[\mathbf{a}] = \frac{d^2 \mathbf{u}}{dt^2} = \frac{[L]^1}{[T]^2}, \text{ m/s}^2$





# Approximation of a function

## ◎ First order of approximation

$$\circ \tilde{f}_1(x) = f(a) + f'(a)(x-a)$$

When  $f(a)$  and  $f'(a)$  are known at  $x=a$

$$\circ f(x) = \tilde{f}_1(x) + O(\varepsilon^2) = f(a) + f'(a)(x-a) + O(\varepsilon^2)$$

## ◎ Second order of approximation

$$\circ \tilde{f}_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

When  $f(a)$ ,  $f'(a)$  and  $f''(a)$  are known at  $x=a$

$$\circ f(x) = \tilde{f}_2(x) + O(\varepsilon^3)$$

## ◎ Third order of approximation

$$\circ \tilde{f}_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3$$

When  $f(a)$ ,  $f'(a)$ ,  $f''(a)$  and  $f'''(a)$  are known at  $x=a$

$$\circ f(x) = \tilde{f}_3(x) + O(\varepsilon^4)$$

## ◎ $n$ -th order of approximation

$$\circ f(x) = \tilde{f}_n(x) + O(\varepsilon^{n+1}) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(a) x^i + O(\varepsilon^{n+1}) \quad \lim_{x \rightarrow a} O(\varepsilon^{(i+1)}) / (x-a)^i = 0$$



# Taylor series

◎ First order of approximation of the function  $f(x)$  at  $x=a+\Delta x$

- $f(a+\Delta x) \approx \tilde{f}_1(a+\Delta x) = f(a) + f'(a)\Delta x$  x  $\Leftarrow a + \Delta x$      $x \Leftarrow x + \Delta x$
- $f(x+\Delta x) \approx \tilde{f}_1(x+\Delta x) = f(x) + f'(x)\Delta x$   $\tilde{f}_1(x) = f(a) + f'(a)(x-a)$

◎  $n$ -th order of approximation of the function  $f(x)$  at  $x=a+\Delta x$

- $f(a+\Delta x) \approx \tilde{f}_n(a+\Delta x) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(a) \Delta x^i$  a  $\Leftarrow x$
- $f(x+\Delta x) \approx \tilde{f}_n(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2!} f''(x)\Delta x^2 + \dots + \frac{1}{n!} f^{(n)}(x)\Delta x^n$  a  $\Leftarrow 0$   
 $\Delta x \Leftarrow x$
- $f(x+\Delta x) \approx \tilde{f}_n(x+\Delta x) = f(x) + \Delta f + \Delta^2 f + \dots + \Delta^n f$   
 $\Delta f \equiv \frac{1}{1!} \frac{df}{dx} \Delta x, \Delta^2 f \equiv \frac{1}{2!} \frac{d^2 f}{dx^2} \Delta x^2, \dots, \Delta^n f \equiv \frac{1}{n!} \frac{d^n f}{dx^n} \Delta x^n$
- $\tilde{f}_n(a+\Delta x), \tilde{f}_n(x+\Delta x)$ :  $n$ -th Taylor series expansion

- $\Delta f = \frac{df}{dx} \Delta x$ : First approximation

- $\lim_{\Delta x \rightarrow 0} \Delta^{n+1} f / \Delta x^n = 0$

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) x^i$$

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \dots$$



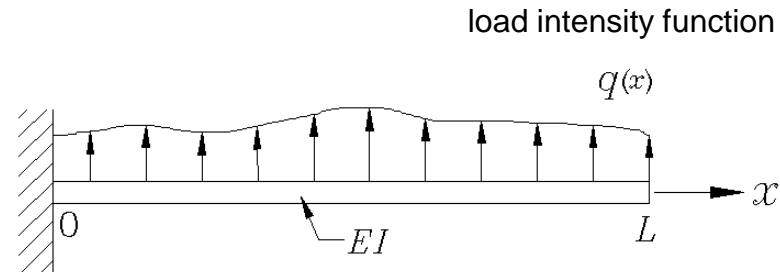
# Ordinary differential equation (ODE)

- ODE: Equation having ordinary derivatives.
- Order of ODE: The maximum order of differentiation of the derivatives in ODE
- Linear and nonlinear ODEs: A linear ODE has all linear terms with respect to functions and derivatives. Otherwise, the DE is a non-linear equation.
  - Example of linear DE:  $y'' + p(x)y' + q(x)y = f(x)$
  - Example of non-linear DE:  $y'' + y'y = f(x)$ ,  $y' + y^2 = f(x)$
- Examples in solid mechanics
  - $\frac{dM_b(x)}{dx} + V(x) = 0$ ,  $\frac{dV(x)}{dx} + q(x) = 0$       Equation, solution = discrete real value  
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
  - $(EIv''(x))'' = M_b'' = q(x)$
- Solution of linear ODE = Homogeneous solution ( $y_h$ ) + particular solution ( $y_p$ )
  - $y_h$ :  $y_h'' + p(x)y_h' + q(x)y_h = 0$
  - $y_p$ : Particular solution which satisfy balance between the right and left of the DE.
  - Homogeneous solution involves unknown constants which are determined by boundary conditions and initial conditions.



# Boundary conditions, boundary value problem

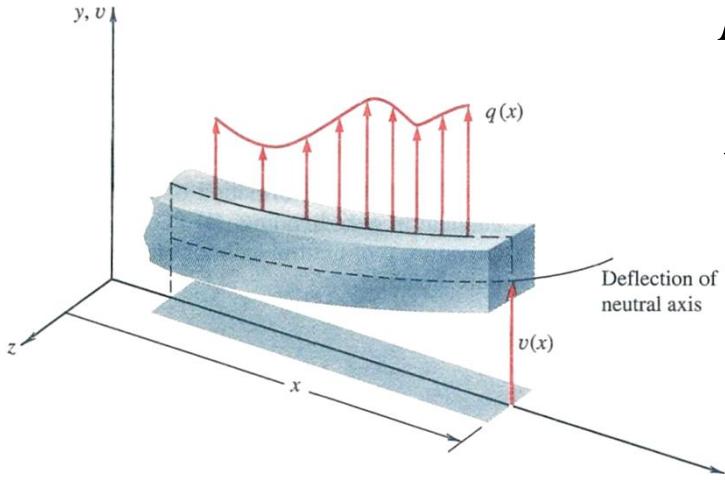
- Boundary Value Problem (BVP) = DE + Boundary conditions (BC)
- Differential equation
  - Order of DEs in mechanics is even, denoted as  $2p$ .
  - Number of unknown constants is equal to order of the DE, which are determined by boundary or initial conditions.
- Classification of boundary conditions
  - Essential BC: Boundary conditions involving the derivatives of order  $0 \sim (p - 1)$ .
  - Natural BC: Boundary conditions involving the derivatives of order  $p \sim (2p - 1)$ .
- Examples
  - DE:  $EI \frac{d^4 v}{dx^4} = q(x)$
  - BC: (Essential BC)  $v(0) = 0, v'(0) = 0$   
(Natural BC)  $M_b(L) = EI v''(L) = 0, V(L) = -EI v'''(L) = 0$





# Ordinary differential equation

- Beam deflection



$$M_b'(x) = -V(x)$$

$$\kappa = v''(x) = \frac{M_b(x)}{EI}$$

$$EIv''(x) = M_b(x)$$

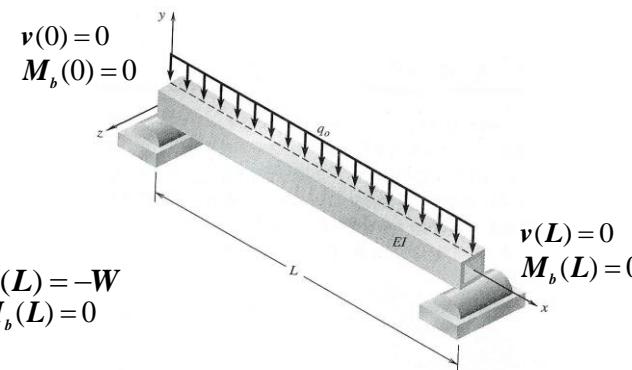
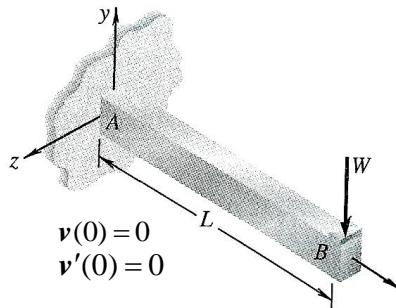
$$(EIv''(x))' = -V(x)$$

$$(EIv''(x))'' = q(x)$$

- Boundary conditions:

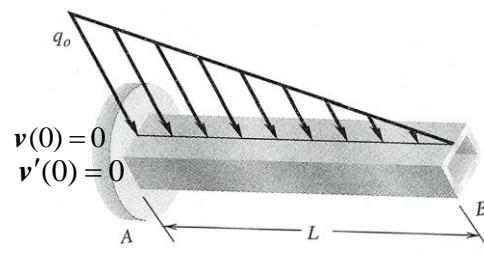
- Essential BC:  $v(0) = 0, v(L) = 0, v'(0) = 0, v'(L) = 0$

- Natural BC:  $V(0) = P, M_b(0) = M, V(L) = P, \dots$



$$M_b(x) = EIv''(x)$$

$$V(x) = -M'_b(x) = -(EIv''(x))'$$

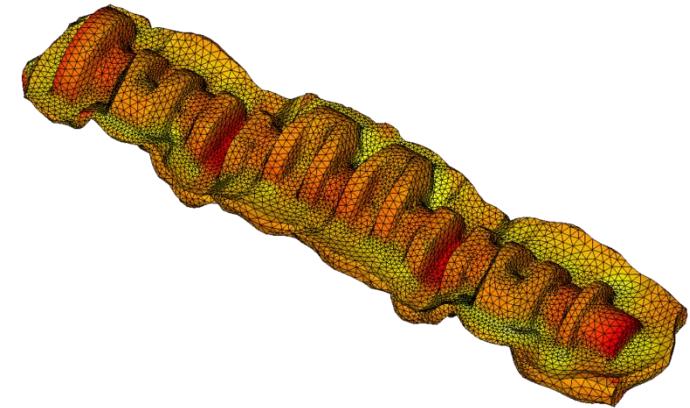




# Multi-variable vector function

## ◎ Multi-variable function

- Function of two or more independent variables
- For two or more dimensional problem



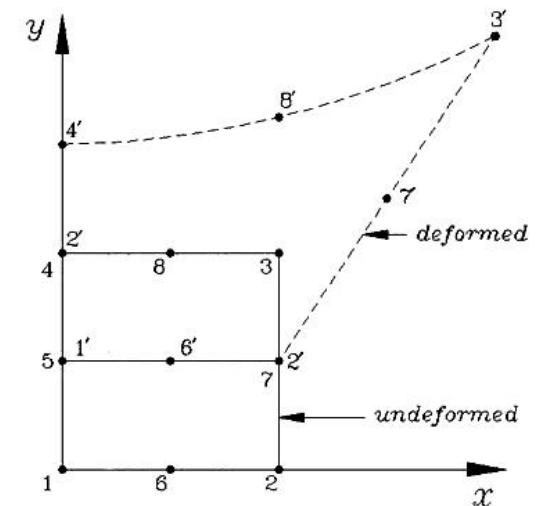
## ◎ Classification of multi-variable function

- Only one dependent variable: Scalar function
- Two or more dependent variables: Vector function

## ◎ Multi-variable function in mechanics

- Two independent variables:
  - Beam deflection curve  $v = v(x, t)$  when exerted load is a function of time, i.e.,  $P = P(t)$
  - In plane strain problem, displacement or velocity is described by  $u_i = u_i(x, y)$  or  $v_i = v_i(x, y)$ , i.e., a function of coordinates.

$$u_x = xy \quad u_y = \frac{1}{2}(x^2 y + 1)$$





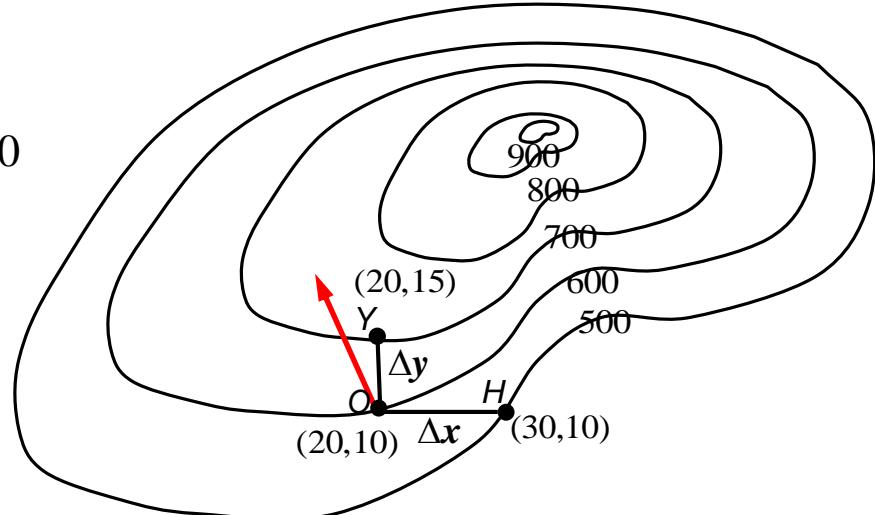
# Partial differentiation and gradient

- Height of a mountain  $h = h(x, y)$

$$\frac{\Delta h}{\Delta x} \Big|_{(x=20, y=10)} = \frac{h(30, 10) - h(20, 10)}{\Delta x} = \frac{500 - 600}{10} = -10$$

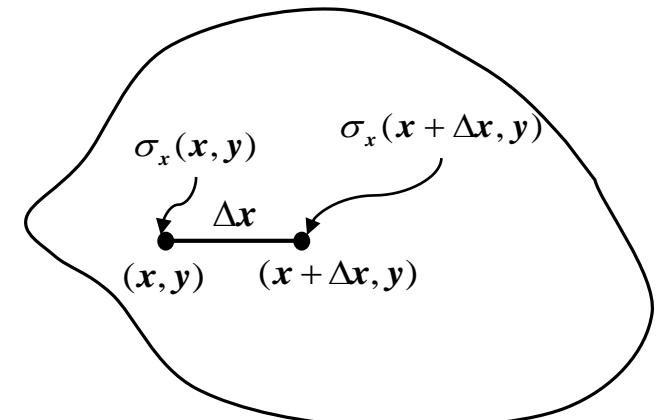
$$\frac{\Delta h}{\Delta y} \Big|_{(x=20, y=10)} = \frac{h(20, 15) - h(20, 10)}{\Delta y} = \frac{700 - 600}{5} = 20$$

$$\frac{\partial h}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{h(x + \Delta x, y) - h(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta h}{\Delta x} \Big|_{y \text{ fixed}}$$



- Stress in 2D  $\sigma_x = \sigma_x(x, y)$

$$\frac{\partial \sigma_x}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\sigma_x(x + \Delta x, y) - \sigma_x(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta \sigma_x}{\Delta x} \Big|_{y \text{ fixed}}$$





# Definition of partial differentiation

## ◎ Average change of multi-variable function

- Average change of multi-variable function in two directions are defined as follows:

$$\frac{\Delta\phi}{\Delta x} \Big|_{y \text{ fixed}} \equiv \frac{\phi(x + \Delta x, y) - \phi(x, y)}{\Delta x}$$

$$\frac{\Delta\phi}{\Delta y} \Big|_{x \text{ fixed}} \equiv \frac{\phi(x, y + \Delta y) - \phi(x, y)}{\Delta y}$$

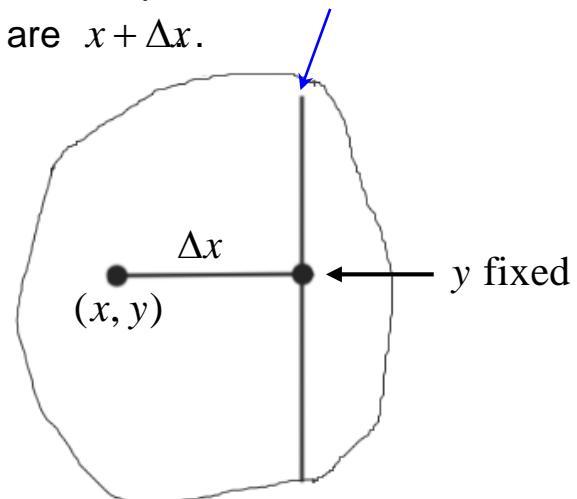
## ◎ Partial differentiation:

$$\frac{\partial\phi}{\partial x} \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta\phi}{\Delta x} \Big|_{y \text{ fixed}} = \lim_{\Delta x \rightarrow 0} \frac{\phi(x + \Delta x, y) - \phi(x, y)}{\Delta x}$$

$$\frac{\partial\phi}{\partial y} \equiv \lim_{\Delta y \rightarrow 0} \frac{\Delta\phi}{\Delta y} \Big|_{x \text{ fixed}} = \lim_{\Delta y \rightarrow 0} \frac{\phi(x, y + \Delta y) - \phi(x, y)}{\Delta y}$$

- Example:

A set of points of which x-coordinates are  $x + \Delta x$ .



- When  $\phi = \phi(x, y)$  is a function of independent variables  $x$  and  $y$ , the following ordinary derivative can not be defined because there are so many points which are apart from  $(x, y)$  by  $\Delta x$  in the  $x$ -direction.

$$\frac{d\phi}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta\phi}{\Delta x}$$



# Gradient of function

- ◎ Definition of gradient of  $\phi = \phi(x, y, z)$

- $\nabla \phi \equiv \left[ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right]^T = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$

- ◎ Differential operator  $\nabla$

- $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$
- $\nabla(f \pm g) = \nabla f \pm \nabla g$
- $\nabla(fg) = g\nabla f + f\nabla g$
- $\nabla(f/g) = \frac{1}{g^2} (g\nabla f - f\nabla g)$

- ◎ Geometric meaning of gradient

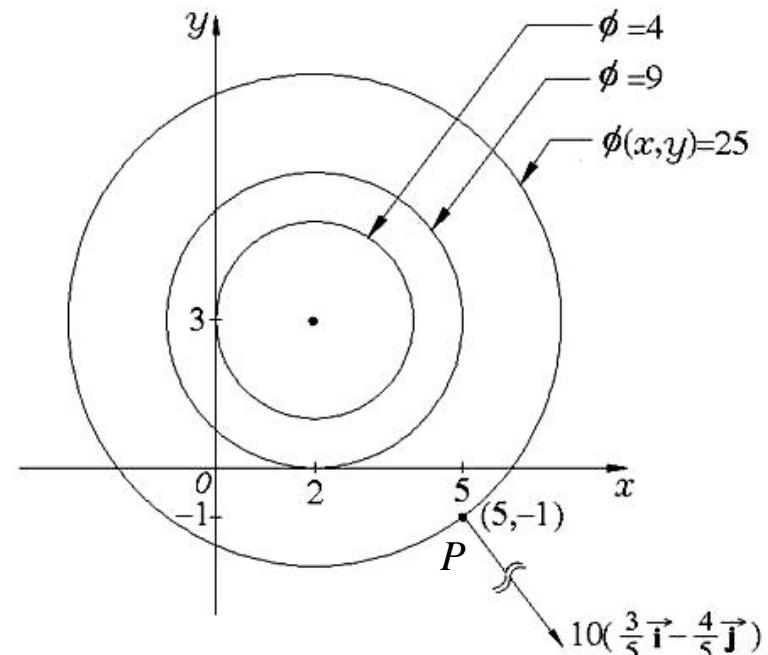
- Example:  $\phi(x, y) = (x - 2)^2 + (y - 3)^2$

- Gradient at  $P(5, -1)$ :  $\nabla \phi = 10 (0.6 \mathbf{i} - 0.8 \mathbf{j})$

- Magnitude: 10, Direction:  $\mathbf{n} = 0.6 \mathbf{i} - 0.8 \mathbf{j}$

- Normal to the contour toward the increasing direction of function

- Steepest descent method: Minimization scheme of function in which a new point is iteratively determined following the negative of gradient.



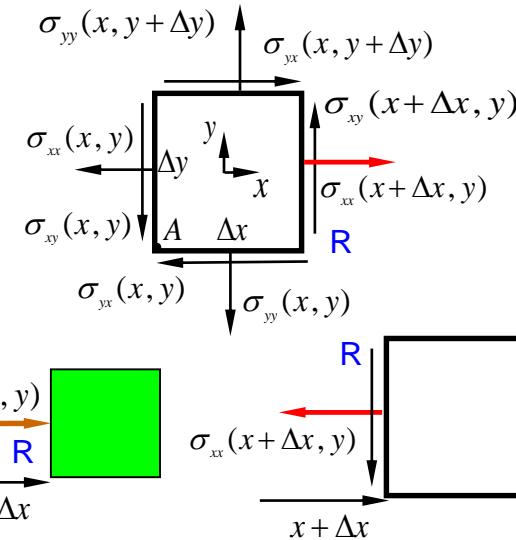
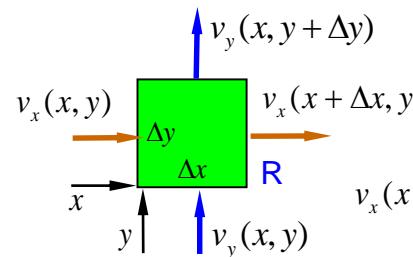


# Taylor series of multi-variable function

◎ First order approximation of  $\phi = \phi(x_1, x_2, \dots, x_n)$

- $\phi(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) = \phi(x_1, x_2, \dots, x_n)$

$$+ \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \Delta x_i + O(\varepsilon^2)$$



◎ Second order approximation of  $\phi = \phi(x_1, x_2, \dots, x_n)$

- $\phi(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) = \phi(x_1, x_2, \dots, x_n) + \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n H_{ij} \Delta x_i \Delta x_j + O(\varepsilon^3)$

Hessian matrix

◎ Total differential,  $d\phi$

- $\Delta \phi = \phi(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - \phi(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \Delta x_i + O(\varepsilon^2)$

- $d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \dots + \frac{\partial \phi}{\partial x_n} dx_n = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} dx_i = \nabla \phi \cdot d\mathbf{r}$

- $d\mathbf{r} = [dx_1, dx_2, \dots, dx_n]^T$



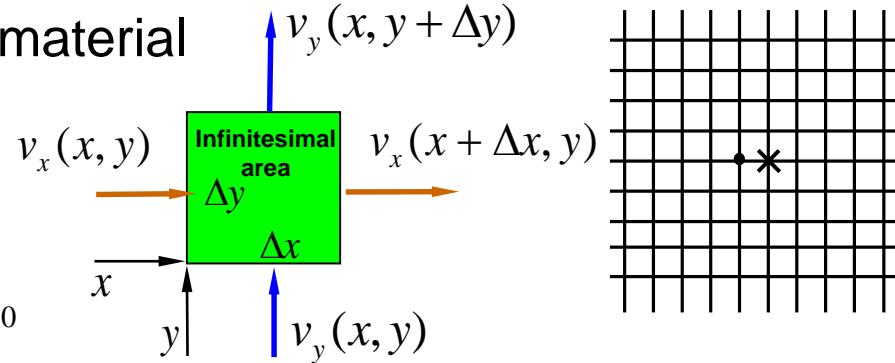
# Partial differential equations

- Continuity equation of incompressible material

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$(v_x(x + \Delta x, y) - v_x(x, y))\Delta y + (v_y(x, y + \Delta y) - v_y(x, y))\Delta x = 0$$

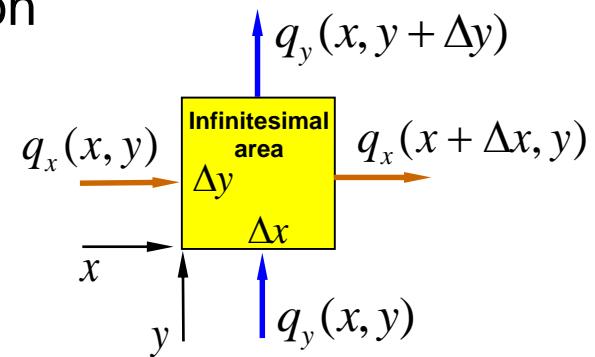
$$(v_x(x + \Delta x, y) - v_x(x, y))/\Delta x + (v_y(x, y + \Delta y) - v_y(x, y))/\Delta y = 0$$



- Equation of conduction under steady-state condition

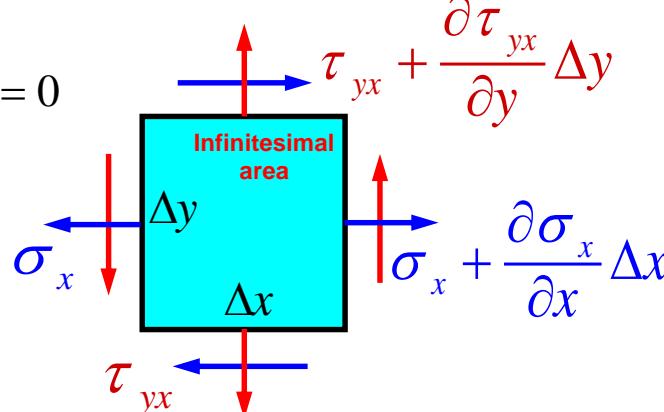
$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \quad \Rightarrow \Rightarrow \Rightarrow \quad \frac{\partial}{\partial x} \left( k \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial \phi}{\partial y} \right) = 0$$

$$q_x = -k \frac{\partial \phi}{\partial x}, \quad q_y = -k \frac{\partial \phi}{\partial y}$$



- Equation of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$



$$q_x(x + \Delta x, y) = q_x(x, y) + \frac{\partial q_x}{\partial x} \Delta x$$

$$q_y(x, y + \Delta y) = q_y(x, y) + \frac{\partial q_y}{\partial y} \Delta y$$

- Miscellaneous

$$\varepsilon_x = \lim_{\Delta x \rightarrow 0} \frac{\Delta u_x}{\Delta x} \Big|_{y \text{ fixed}} = \frac{\partial u_x}{\partial x}$$

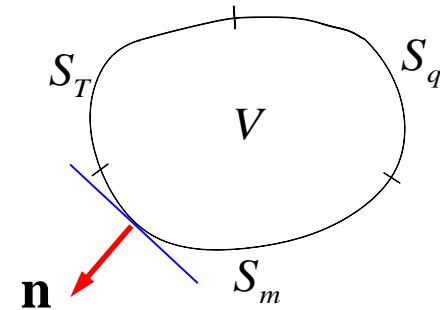


# Boundary value problem – Heat conduction

## ◎ Partial differential problem

$$\frac{\partial}{\partial x} \left( k \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial \phi}{\partial z} \right) + q_g = \rho c \frac{\partial \phi}{\partial t}$$

$$(k\phi_i)_{,i} + q_g = \rho c \frac{\partial \phi}{\partial t}$$



## ◎ Boundary conditions

- Dirichlet condition:  $\phi = \bar{\phi}$  on  $S_T$  ← Essential BC
- Neumann(flux) condition:  $\frac{\partial \phi}{\partial n} = (\nabla \phi \cdot \mathbf{n}) = \bar{q}$  on  $S_q$  ← Natural BC
- Robin or mixed condition:  $\alpha \phi + \beta \frac{\partial \phi}{\partial n} = g$  on  $S_m$  ← Natural/Essential BC

- $\alpha, \beta, g, \bar{\phi}, \bar{q}$ : Functions of their related boundaries or constants
- $\mathbf{n}$ : Outwardly directed normal unit vector
- Surface definition:  $S = S_\phi \cup S_q \cup S_m$ ,  $\emptyset = S_\phi \cap S_q$ ,  $\emptyset = S_\phi \cap S_m$ ,  $\emptyset = S_q \cap S_m$



# Boundary value problem – Elasticity

◎ Partial differential problem

$$\sigma_{ij,j} + f_i = 0 \text{ in } V$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - (3\lambda + 2\mu)a\Delta T\delta_{ij}$$

Indicial notation

◎ Boundary conditions

$$u_i = \bar{u}_i \quad \text{on } S_{u_i}$$

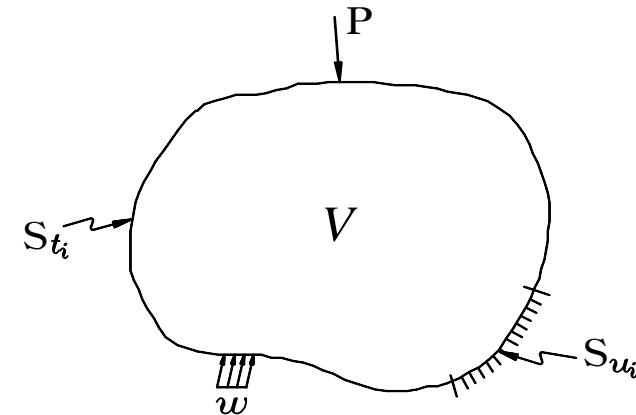
$$t_i^{(n)} = \bar{t}_i \quad \text{on } S_{t_i}$$

$$\alpha u_i + \beta t_i^{(n)} = g \quad \text{on } S_{m_i}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$



$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

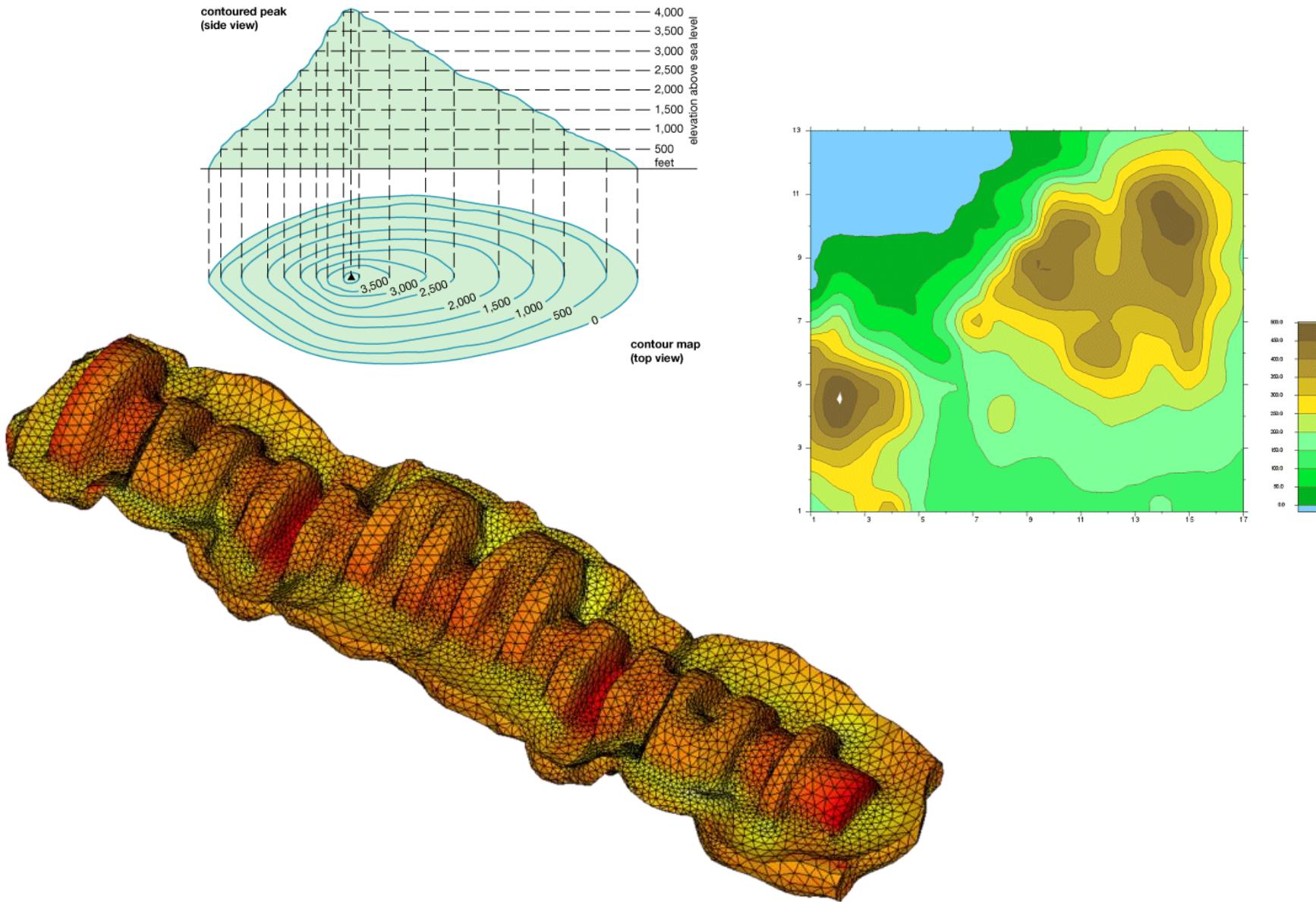
$$\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \text{ etc.}$$

$$\left. \begin{aligned} \varepsilon_x &= \frac{1}{E} [\sigma_x - v(\sigma_y + \sigma_z)] + \alpha \Delta T \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - v(\sigma_x + \sigma_z)] + \alpha \Delta T \\ \varepsilon_z &= \frac{1}{E} [\sigma_z - v(\sigma_x + \sigma_y)] + \alpha \Delta T \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy}, \gamma_{yx} = \frac{1}{G} \tau_{yz}, \gamma_{zx} = \frac{1}{G} \tau_{zx} \end{aligned} \right\}$$

$$S = S_{u_i} \cup S_{t_i} \cup S_{m_i}, \quad \phi = S_{u_i} \cap S_{t_i}, \quad \phi = S_{u_i} \cap S_{m_i}, \quad \phi = S_{t_i} \cap S_{m_i} \quad (i = 1, 2, 3)$$



# Discrete description of a complicated function





# Extremization of one variable scalar function

## ○ Extremization of $y = f(x)$

$$\textcircled{C} \quad f(x^* + \varepsilon) = f(x^*) + \varepsilon f'(x^*) + \frac{1}{2!} \varepsilon^2 f''(x^*) + \frac{1}{3!} \varepsilon^3 f^{(3)}(x^*) + \frac{1}{4!} \varepsilon^4 f^{(4)}(x^*) + \dots :$$

### ○ Local minimization

- $f(x^* + \varepsilon) - f(x^*) = \varepsilon f'(x^*) + \frac{1}{2!} \varepsilon^2 f''(x^*) + \frac{1}{3!} \varepsilon^3 f^{(3)}(x^*) + \frac{1}{4!} \varepsilon^4 f^{(4)}(x^*) + \dots > 0$
- $f(x^*) \leq f(x) = f(x^* + \varepsilon)$ : Condition for extremum point at  $x = x^*$ 
  - $f(x^*) = 0$  and  $f''(x^*) > 0$
  - $f'(x^*) = f''(x^*) = \dots = f^{(2n-1)}(x^*) = 0$  and  $f^{(2n)}(x^*) > 0$
  - $\varepsilon$ : real number of which absolute is sufficiently small.

$$f(x^*) \geq f(x^* + \varepsilon) : x = x^*$$

### ○ Local maximization

- $f(x^*) \geq f(x^* + \varepsilon)$ : Condition for extremum point at  $x = x^*$ 
  - $f(x^*) = 0$  and  $f''(x^*) < 0$
  - $f'(x^*) = f''(x^*) = \dots = f^{(2n-1)}(x^*) = 0$  and  $f^{(2n)}(x^*) < 0$

### ○ Conditions

- $f'(x^*) = 0$  : **Necessary condition for an extremum point,**  
**Necessary and sufficient condition for a stationary point which may be an extremum point or inflection point.**

$$y = x^4, y' = 4x^3, y'' = 12x^2, y''' = 24x$$



# Unconstrained extr. of multi-variable function

- Extremization of  $\phi = \phi(x_1, x_2, \dots, x_n) = \phi(\mathbf{x})$  at  $\mathbf{x} = \mathbf{x}^*$
- $\phi(x_1^*, x_2^*, \dots, x_n^*) \geq \phi(x_1^* + \varepsilon_1, x_2^* + \varepsilon_2, \dots, x_n^* + \varepsilon_n)$ 
  - $\boldsymbol{\varepsilon}^T = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]$ : Arbitrary vector of which magnitude is sufficiently small.
  - Necessary condition:  $\frac{\partial \phi}{\partial x_i} \Big|_{x=x^*} = 0, \quad i=1, 2, \dots, n$
  - Sufficient condition for local maximization at  $\mathbf{x} = \mathbf{x}^*$  :  
Hessian matrix  $\mathbf{H}$  should be negative-definite
    - $\frac{1}{2} \boldsymbol{\varepsilon}^T \mathbf{H}(x^*) \boldsymbol{\varepsilon} + O(\|\boldsymbol{\varepsilon}\|^3) \leq 0$
    - $\phi(x_1^* + \varepsilon_1, x_2^* + \varepsilon_2, \dots, x_n^* + \varepsilon_n) = \phi(x_1^*, x_2^*, \dots, x_n^*) + \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n H_{ij} \varepsilon_i \varepsilon_j + O(\|\boldsymbol{\varepsilon}\|^3)$
  - Sufficient condition for local minimization at  $\mathbf{x} = \mathbf{x}^*$  :  
Hessian matrix  $\mathbf{H}$  should be positive-definite



# Lagrange's multiplier method for extremization

## ◎ An example of minimization problem

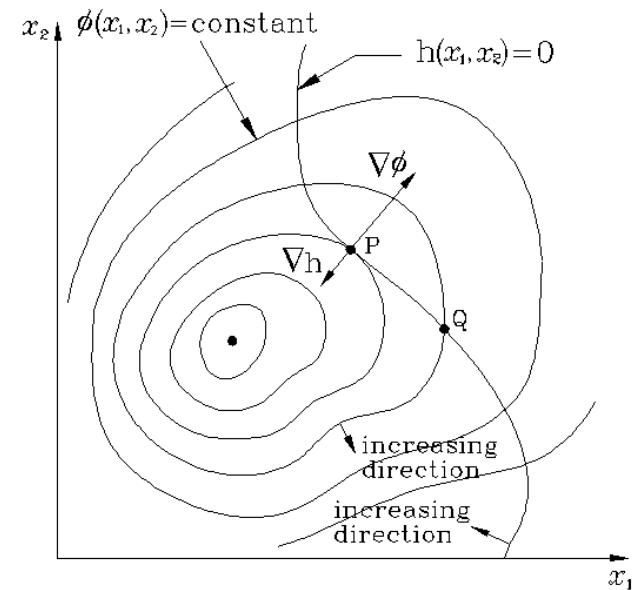
*Minimize*  $\phi = \phi(x_1, x_2)$

*subject to*  $h(x_1, x_2) = 0$

## ○ Condition for solution

$$\nabla \phi = c \nabla h, \lambda = -c$$

- Directions of gradient should be identical.
- $\nabla \phi(x_1, x_2) + \lambda \nabla h(x_1, x_2) = 0$
- $h(x_1, x_2) = 0$ 
  - $\lambda$  : Lagrange's multiplier



<Constrained minimization>

## ○ Necessary condition of a new function, $\Lambda = \Lambda(x_1, x_2, \lambda) = \phi(x_1, x_2) + \lambda h(x_1, x_2)$

- $\nabla_x \Lambda = 0$  (or  $\nabla \phi(x_1, x_2) + \lambda \nabla h(x_1, x_2) = 0$ )
- $\nabla_\lambda \Lambda = 0$  (or  $h(x_1, x_2) = 0$ )
  - $\lambda, x_1, x_2$  : Independent variables
  - $\nabla_x, \nabla_\lambda$  : Gradients of variable  $x=(x_1, x_2)$  and  $\lambda$ , respectively

$$\frac{\partial \Lambda}{\partial x_1} = \frac{\partial \phi}{\partial x_1} + \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial \Lambda}{\partial x_2} = \frac{\partial \phi}{\partial x_2} + \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = h(x_1, x_2) = 0$$



# Generalization of Lagrange's multiplier method

- Extremization problem with equality constraints

*Extremize  $\phi = \phi(x_1, x_2, \dots, x_n)$*

*subject to  $h_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, m$*



- Unconstrained extremization problem

*Extremize  $\Lambda(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m)$*

$$= \phi(x_1, x_2, \dots, x_n) + \sum_{i=1}^m \lambda_i h_i(x_1, x_2, \dots, x_n)$$

- Necessary conditions

- $\nabla_{x_i} \Lambda(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = 0, i = 1, 2, \dots, n$
- $\nabla_{\lambda_i} \Lambda(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = 0, i = 1, 2, \dots, m$

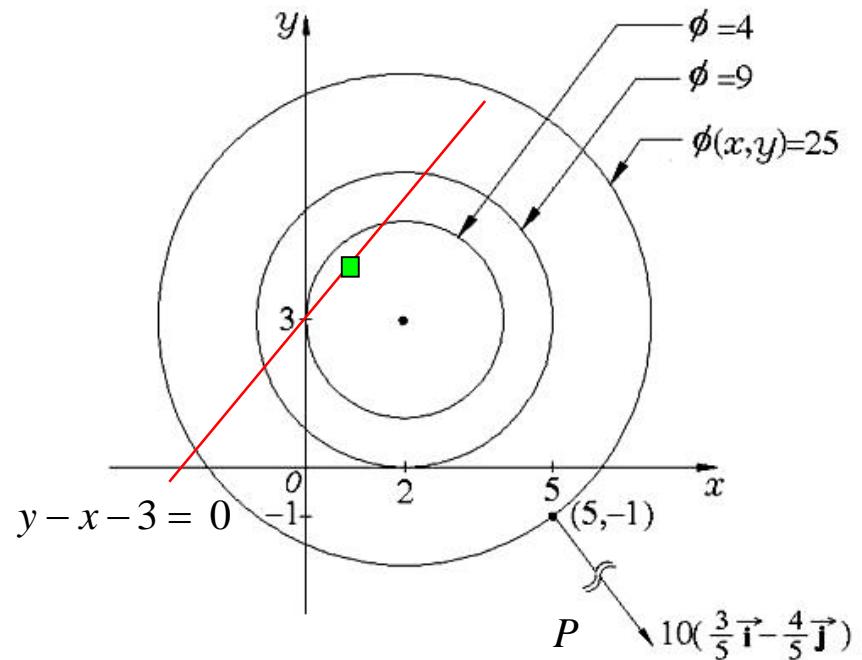


# Lagrange's multiplier method for extremization

◎ An example of minimization problem

$$\text{Minimize } \phi(x, y) = (x - 2)^2 + (y - 3)^2$$

$$\text{subject to } y - x - 3 = 0$$



$$\text{Extremize } \phi(x, y, \lambda) = (x - 2)^2 + (y - 3)^2 + \lambda(y - x - 3)$$

$$2(x - 2) - \lambda = 0$$

$$2(y - 3) + \lambda = 0 \Rightarrow x = 1, y = 4, \lambda = -4$$

$$y - x - 3 = 0$$



# Extremization using penalty method

- Extremization problem with equality constraints

*Extremize*  $\phi = \phi(x_1, x_2, \dots, x_n)$

*subject to*  $h_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, m$



- Unconstrained extremization problem

*Extremize*  $P(x_1, x_2, \dots, x_n) = \phi(x_1, x_2, \dots, x_n) + \sum_{i=1}^m P_i h_i^2(x_1, x_2, \dots, x_n)$

- $P_i$ : Penalty constant

- Necessary conditions:  $\nabla_{x_i} P(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, n$



## 2.4 Integration



# Lebesgue integral and Riemann integral

## ○ Integration:

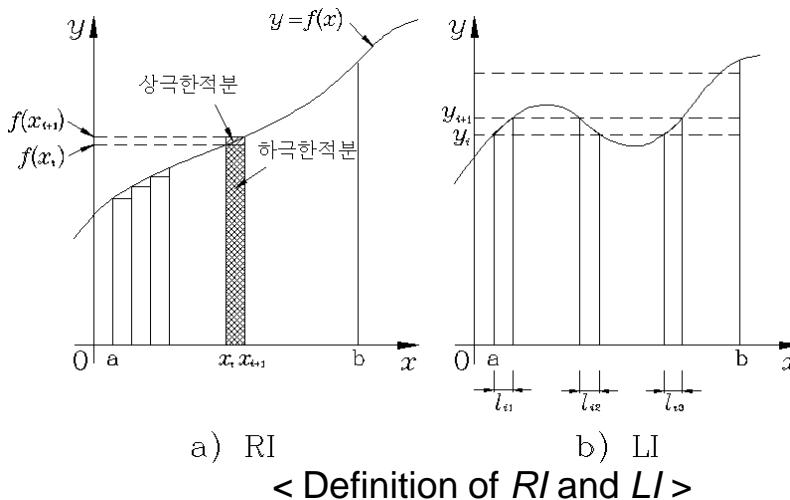
- Definite integration: Integral with its integration region specified

- $\int_a^b f(x)dx, \int_A \phi(x, y)dA, \int_V \phi(x, y, z)dV$

- Indefinite integration: Integral with its integration region unspecified

- $\int f(x)dx, \int \phi(x, y)dA, \int \phi(x, y, z)dV$

## ○ Riemann integral(RI) and Lebesgue integral(LI)



- $RI = \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_{i+1} - x_i) f(x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_{i+1} + x_i) f(x_{i+1})$

- When upper limit integral should be identical to lower limit integral, RI exists.

- $LI = \lim_{n \rightarrow \infty} \sum_{i=1}^n y_i \sum_{k=1}^{l_{ik}} l_{ik} = \lim_{n \rightarrow \infty} \sum_{i=1}^n y_{i+1} \sum_{k=1}^{l_{ik}} l_{ik}$

- When the above requirement meets, LI exists.

- LI is defined whenever RI is defined. However, RI can not be always defined when LI is defined.

- Example: Mixed liquid of 1kg of water and 1kg of alcohol.



# Line integration and its standard form

## ○ Line integration

- When  $|f(x)| \leq K$   $\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx$

- $\int_a^b f(x)dx = \int_a^b f(\xi)d\xi$  ←  $x, \xi$  are dummy parameters

- $\int_a^b u'v dx = uv \Big|_a^b - \int_a^b u v' dx$  ←  $\int_a^b (uv)' dx = uv \Big|_a^b = \int_a^b (u v' + u'v) dx$

- When  $f(x) = dg/dx$ , the integral can be calculated using function values at the integration boundaries as follow:

$$\int_a^b f(x) dx = \int_a^b \frac{dg(x)}{dx} dx = g(x) \Big|_a^b = g(b) - g(a)$$

- Standard form  $\int_{-1}^1 f(x(\xi)) \frac{dx}{d\xi} d\xi$

- $\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(x(\xi)) \frac{dx}{d\xi} d\xi$  ↑  $x = x(\xi)$

$$x = x(\xi) = N_1(\xi)a + N_2(\xi)b \quad N_1(\xi) = \frac{1-\xi}{2}, \quad N_2(\xi) = \frac{1+\xi}{2}$$

$$\alpha = -1, \quad \beta = 1$$

$$x = x(\xi) = \frac{1-\xi}{2}a + \frac{1+\xi}{2}b$$

$$\begin{aligned} x &= 2t, \quad dx = 2dt \\ x = 4 &\rightarrow t = 2, \quad x = 2 \rightarrow t = 1, \\ \int_2^4 4x^2 dx &= \int_1^2 4(2t)^2 \times 2dt \\ &= \int_{-1}^1 4(\xi+3)^2 d\xi = 4\left(\frac{1}{\sqrt{3}}+3\right)^2 + 4\left(\frac{-1}{\sqrt{3}}+3\right)^2 \\ x &= 4 \times \frac{1+\xi}{2} + 2 \times \frac{1-\xi}{2} = \xi+3, \quad dx = d\xi \end{aligned}$$

$$\int_a^b f(x)dx = \int_{-1}^1 f(x(\xi)) J d\xi$$

- $J = \frac{b-a}{2}$  ← Jacobian



# Path integral

## ○ Path integral: Integration following the path C

$$\textcircled{O} \quad W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_x dx + F_y dy + F_z dz)$$

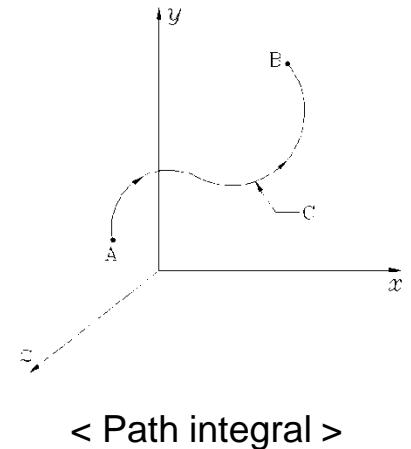
- $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$

- $\oint_C$ : It is used instead of  $\int_C$  when the path C is closed.

$$\textcircled{O} \quad W = \int_{t_1}^{t_2} (F_x(t)x'(t) + F_y(t)y'(t) + F_z(t)z'(t)) dt$$

- $\mathbf{r} = \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$

- $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} = x' dt \mathbf{i} + y' dt \mathbf{j} + z' dt \mathbf{k}$



< Path integral >

## ○ Path independent integral: Path integral which can be calculated only by function values at both ends of the path

- $\mathbf{F} = \nabla \Phi \quad \text{or} \quad F_x = \frac{\partial \Phi}{\partial x}, \quad F_y = \frac{\partial \Phi}{\partial y}, \quad F_z = \frac{\partial \Phi}{\partial z}$

- $\Phi = \Phi(x, y, z)$ : Potential

- Total differential:  $d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$

- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_B^A d\Phi = \Phi(A) - \Phi(B)$

- $\nabla \times \mathbf{F} = 0 \quad \frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$

- When potential  $\Phi$  is defined such that  $\mathbf{F} = \nabla \Phi, \quad \nabla \times \nabla \Phi = \mathbf{0}$ .



# Area integration

- $\int_A \phi(x, y) dA$  or  $\int_A \phi(x, y) dx dy$

$$J \equiv \frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

- Transformation of integration area

- $\int_A \phi(x, y) dx dy = \int_{A'} \phi(x(\xi, \eta), y(\xi, \eta)) J d\xi d\eta$

$x = x(\xi, \eta)$ ,  $y = y(\xi, \eta)$  : Transformation functions which meet one-to-one corresponding.

- Standard form for integration over a quadrilateral

- $\int_A \phi(x, y) dx dy = \int_{-1}^1 \int_{-1}^1 \phi(x(\xi, \eta), y(\xi, \eta)) J d\xi d\eta$

$$x = \sum_{i=1}^4 N_i(\xi, \eta) x_i, \quad y = \sum_{i=1}^4 N_i(\xi, \eta) y_i$$

$$N_i(\xi, \eta) = \frac{(1 + \xi_i \xi)(1 + \eta_i \eta)}{4}$$

$$\xi_1 = \xi_2 = \eta_2 = \eta_3 = 1, \quad \xi_3 = \xi_4 = \eta_1 = \eta_2 = -1$$

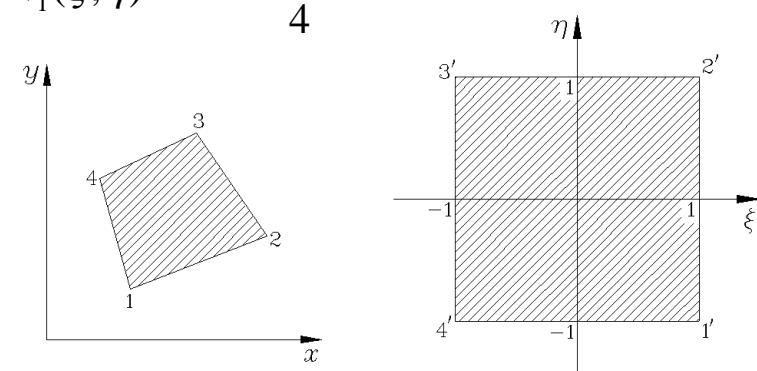
- Four internal corner angles should be less than  $180^\circ$ .

- $(x_i, y_i)$ : Four corner points of a quadrilateral

$$x = x(\xi) = \frac{1 - \xi}{2} a + \frac{1 + \xi}{2} b$$

$$N_1(\xi) = \frac{1 - \xi}{2}, \quad N_2(\xi) = \frac{1 + \xi}{2}$$

$$N_1(\xi, \eta) = \frac{(1 + \xi)(1 - \eta)}{4}$$



< Integration region transformation from quadrilateral to square>

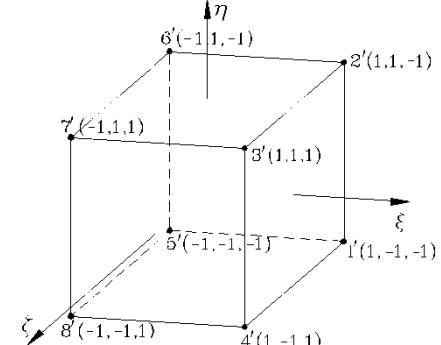
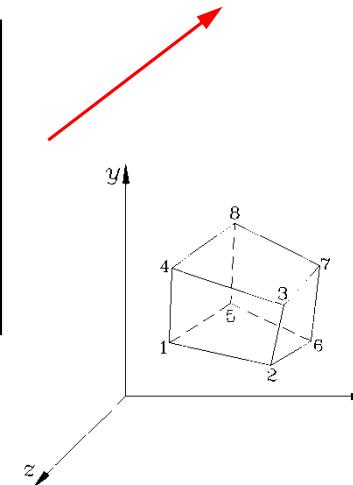


# Volume integration

$$\textcircled{O} \int_V \phi(x, y, z) dx dy dz = \int_{V'} \phi(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) J d\xi d\eta d\zeta$$

$$\begin{aligned}x &= x(\xi, \eta, \zeta) \\y &= y(\xi, \eta, \zeta) \\z &= z(\xi, \eta, \zeta)\end{aligned}$$

$$J \equiv \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}$$



< Transformation from hexahedral to cube >

O Standard form: (-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)

$$\textcircled{1) } \bullet \int_V \phi(x, y, z) dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \phi(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) J d\xi d\eta d\zeta$$

$$x = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) x_i, \quad y = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) y_i \quad z = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) z_i$$

$$\bullet \quad N_i(\xi, \eta, \zeta) = \frac{(1 + \xi_i \xi)(1 + \eta_i \eta)(1 - \zeta_i \zeta)}{8}$$

$$\bullet \quad \begin{aligned}\xi_1 &= \xi_2 = \xi_3 = \xi_4 = \eta_2 = \eta_3 = \eta_6 = \eta_7 = \zeta_3 = \zeta_4 = \zeta_7 = \zeta_8 = 1 \\ \xi_5 &= \xi_6 = \xi_7 = \xi_8 = \eta_1 = \eta_4 = \eta_5 = \eta_8 = \zeta_1 = \zeta_2 = \zeta_5 = \zeta_6 = -1\end{aligned}$$

$$N_1(\xi, \eta, \zeta) = \frac{(1 + \xi)(1 - \eta)(1 - \zeta)}{8}$$



# Green theorem and Gauss theorem

## ◎ Green theorem

$$\circ \int_A \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy = \oint_C (F_x dx + F_y dy)$$

## ○ Example

- $A = \int_A dx dy = \oint_C x dy = - \oint_C y dx$
- $V = 2\pi \int_A x dx dy = \pi \oint_C x^2 dy$

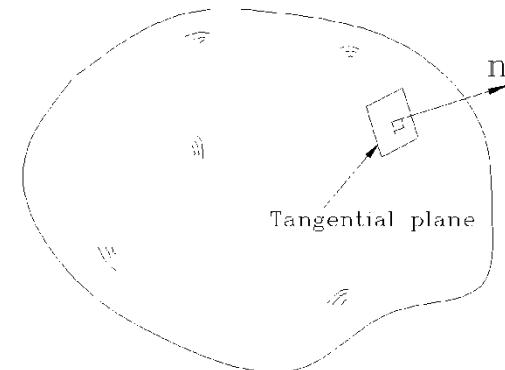
## ◎ Gauss theorem(divergence Theorem) :

$$\circ \int_V \nabla \cdot \mathbf{F} dV = \int_V \operatorname{div} \mathbf{F} dV = \int_S \mathbf{F} \cdot \mathbf{n} dS \quad \text{or}$$

$$\int_V \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV = \int_S (F_x n_x + F_y n_y + F_z n_z) dS$$

- $\mathbf{n}$  : outwardly directed unit normal vector on the boundary

$$\int_a^b \frac{dg(x)}{dx} dx = g(x)|_a^b = g(b) - g(a)$$



< outwardly directed unit normal  
vector on the boundary >