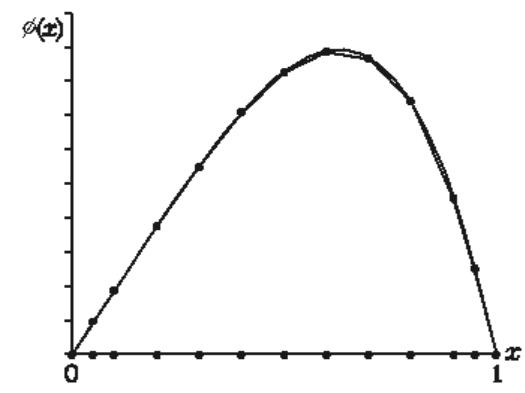
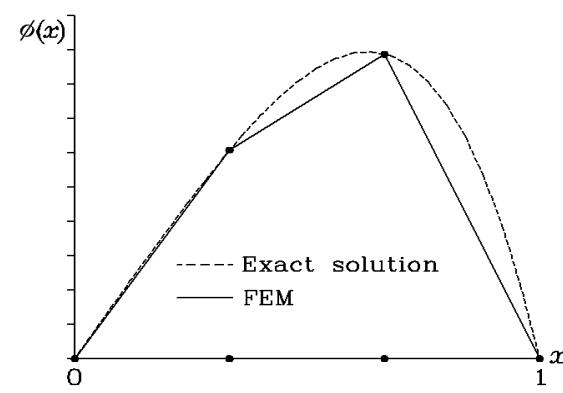
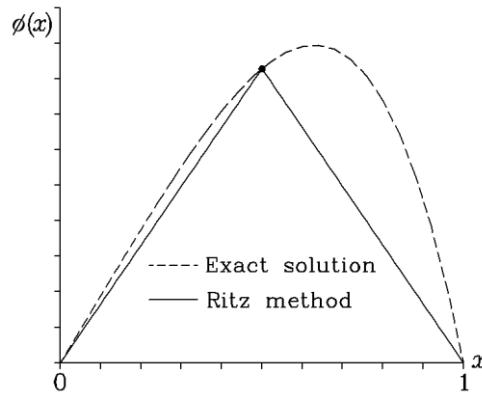


V. Finite Element Method

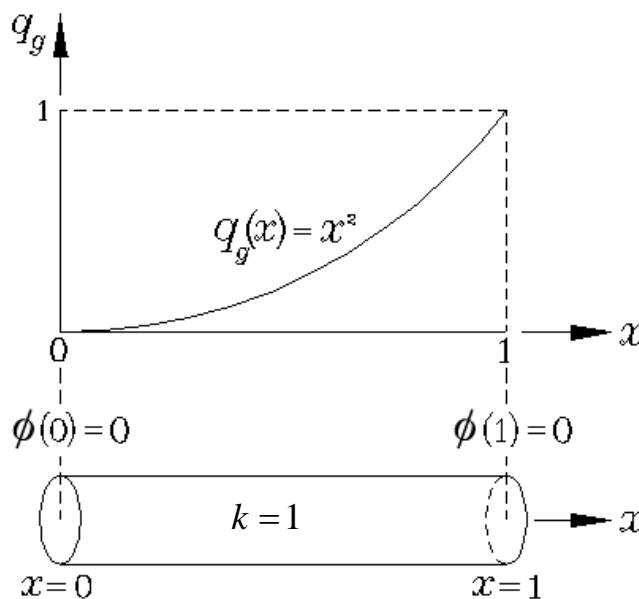
5.1 Introduction to Finite Element Method

5.1 Introduction to FEM



Ritz method to differential equation

- Problem definition



- Boundary value problem

Prob. 1

$$-\frac{d^2\phi}{dx^2} = x^2, \quad 0 < x < 1$$

$$\phi(0) = 0, \phi(1) = 0$$

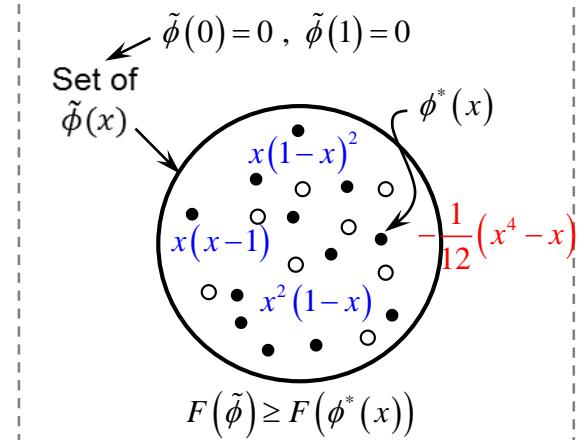
$$\phi''(x) = -x^2 \Rightarrow \phi'(x) = -\frac{1}{12}x^4 + C_1x + C_2$$

- Exact : $\phi^*(x) = -\frac{1}{12}(x^4 - x)$

- Variational principle

Prob. 2 Extremize $F(\phi) = \frac{1}{2} \int_0^1 \left[\left(\frac{d\phi}{dx} \right)^2 - 2x^2 \phi(x) \right] dx$

subject to $\phi(0) = 0, \phi(1) = 0$

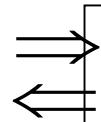




Ritz method to differential equation

- Extremization of a function and its related algebraic equation

$$y' = x(x-1)(2x-1) = 0$$



Extremize $y = x^2(x-1)^2$

- Extremization of a functional and its related differentiation equation

$$-\frac{d^2T}{dx^2} = x^2, \quad 0 < x < 1, \quad T(0) = 0, \quad T(1) = 0$$

(\equiv Prob. A)



$$\begin{aligned} \text{Extremize } F(T) &= \int_0^1 \left[\left(\frac{dT}{dx} \right)^2 - 2x^2 T(x) \right] dx \\ T(0) &= 0, \quad T(1) = 0 \end{aligned}$$

(\equiv Prob. B)

✓ Trial function: $\tilde{T}(x) = \sum_{i=1}^n C_i f_i(x), \quad \tilde{T}(0) = 0, \quad \tilde{T}(1) = 0$



Ritz method to differential equation

- **Approximation**

$$\text{Trial function} \quad \tilde{\phi}(x) = C_1 x (1-x) + C_2 x^2 (1-x)$$

- Transformation: Function space \Rightarrow Finite dimensional vector space

$$F(\tilde{\phi}) = \frac{1}{2} \int_0^1 \left[\left\{ C_1(1-2x) + C_2(2x-3x^2) \right\}^2 - 2C_1 x^3 (1-x) - 2C_2 x^4 (1-x) \right] dx$$

$$= \frac{1}{2} \int_0^1 (1-2x)^2 dx \cdot C_1^2 + \frac{1}{2} \int_0^1 (2x-3x^2)^2 dx \cdot C_2^2$$

$$+ \int_0^1 (1-2x)(2x-3x^2) dx \cdot C_1 C_2 - \int_0^1 x^3 (1-x) dx C_1 - \int_0^1 x^4 (1-x) dx C_2$$

$$\circ \tilde{F}(C_1, C_2) \equiv F(\tilde{\phi}) = \frac{1}{6} C_1^2 + \frac{1}{15} C_2^2 + \frac{1}{6} C_1 C_2 - \frac{1}{20} C_1 - \frac{1}{30} C_2$$

$$\circ \text{Necessary condition for } \tilde{F}(C_1, C_2) \text{ to be extreme : } \frac{\partial \tilde{F}}{\partial C_1} = 0, \quad \frac{\partial \tilde{F}}{\partial C_2} = 0$$

$$\circ \text{Linear equation: } \frac{1}{30} \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow C_1 = \frac{1}{15}, C_2 = \frac{1}{6}$$

$$\circ \text{Approximate solution: } \tilde{\phi}(x) = \frac{1}{30} (2x + 3x^2 - 5x^3) \quad \leftrightarrow \quad \text{Exact solution} \quad \phi^*(x) = -\frac{1}{12} (x^4 - x)$$

Extremize $F(\phi) = \frac{1}{2} \int_0^1 \left[\left(\frac{d\phi}{dx} \right)^2 - 2x^2 \phi(x) \right] dx$

subject to $\phi(0) = 0, \phi(1) = 0$

- **Definite integral**

- $\int_0^1 x dx = \frac{1}{2}$

- **Indefinite integral**

- $\int x dx = \frac{x^2}{2}$



Weighted residual approach to differential equation

Prob. 1

$$\frac{d^2\phi}{dx^2} + x^2 = 0, \quad 0 < x < 1$$

$$\phi(0) = 0, \quad \phi(1) = 0$$

$\omega(x)$ is arbitrary.

Prob. 2

$$\int_0^1 \omega(x) \left[\frac{d^2\phi}{dx^2} + x^2 \right] dx = 0$$

$\phi(0) = \phi(1) = 0$ Set of functions

$\omega(x)$: Weighting function

$$\int_a^b u'v = uv \Big|_a^b - \int_a^b uv'$$

$$\phi'(x)\omega(x) \Big|_0^1 - \int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0$$

$$\phi(0) = \phi(1) = 0$$

Assume $\omega(0) = \omega(1) = 0$

Weak form

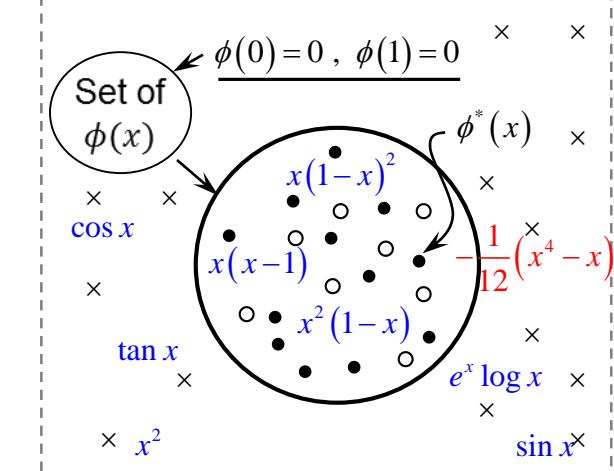
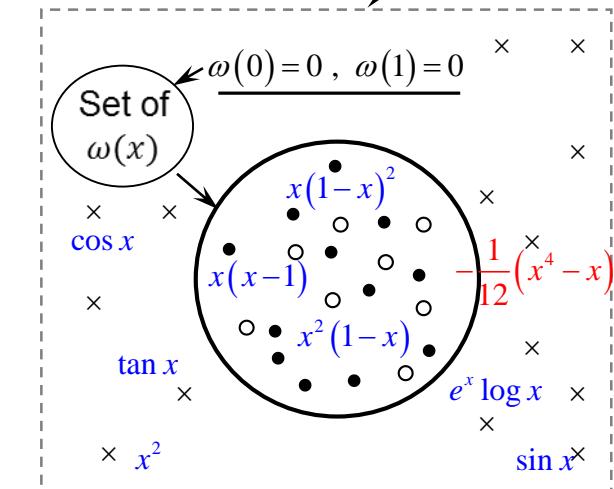
$\phi(x)$: Unknown function

$$\left[\int_0^1 [\phi'(x)\omega'(x) - x^2\omega(x)] dx = 0 \right]$$

$$\phi(0) = 0, \quad \phi(1) = 0$$

where $\omega(x)$ is arbitrary except that

$$\omega(0) = 0 \text{ and } \omega(1) = 0$$





Galerkin approach

○ Approximation of $\phi(x)$ and $\omega(x)$

- $\tilde{\phi}(x) = C_1x(1-x) + C_2x^2(1-x)$; C_1 and C_2 are unknown

Trial function

- $\omega(x) = W_1x(1-x) + W_2x^2(1-x)$; W_1 and W_2 are arbitrary

Approximate weighting function

$$\int_0^1 \left[\{C_1(1-2x) + C_2(2x-3x^2)\} \{W_1(1-2x) + W_2(2x-3x^2)\} dx - W_1x^3(1-x) - W_2x^4(1-x) \right] dx = 0$$

$$\begin{aligned} & W_1 \left[\int_0^1 (1-2x)(1-2x)dx \underset{\downarrow}{C_1} + \int_0^1 (1-2x)(2x-3x^2)dx \underset{\downarrow}{C_2} - \int_0^1 x^3(1-x)dx \right] \\ & + W_2 \left[\int_0^1 (2x-3x^2)(1-2x)dx \underset{\uparrow}{C_1} + \int_0^1 (2x-3x^2)(2x-3x^2)dx \underset{\uparrow}{C_2} - \int_0^1 x^4(1-x)dx \right] = 0 \\ & W_1\Phi_1 + W_2\Phi_2 = 0 \end{aligned}$$

$\Phi_1 \equiv \int_0^1 (1-2x)(1-2x)dx$, $\Phi_2 \equiv \int_0^1 (2x-3x^2)(2x-3x^2)dx$

$\int_0^1 x^5 dx = \frac{1}{6}$, $\int_0^1 x^4 dx = \frac{1}{5}$, etc.

W_1 are W_2 are arbitrary

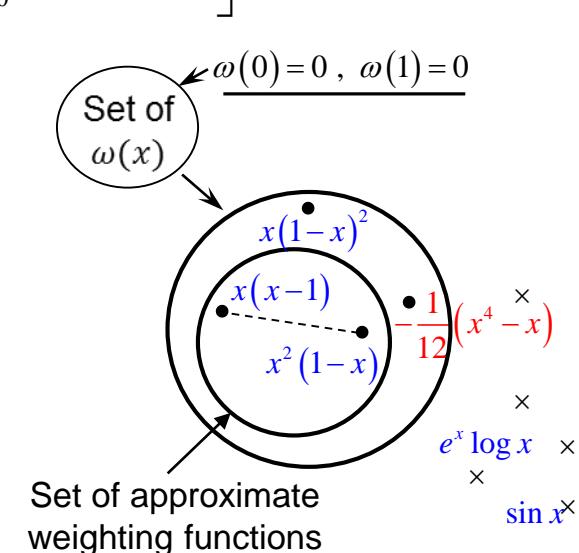
○ Linear equations

$$\begin{aligned} \bullet \Phi_1 = 0 \rightarrow \quad & \frac{1}{30} \begin{bmatrix} 10 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow C_1 = \frac{1}{15}, C_2 = \frac{1}{6} \\ \bullet \Phi_2 = 0 \rightarrow \quad & \end{aligned}$$

○ Approx. solution : • $\tilde{\phi}(x) = \frac{1}{30}(2x + 3x^2 - 5x^3)$

Basic function

$$\begin{cases} \varsigma_1(x) = x(1-x) \\ \varsigma_2(x) = x^2(1-x) \end{cases}$$





Accuracy of the approximate solution

- Comparison between approximate and exact solutions

- $\phi^*(x_{\max}) = 0.029165, \tilde{\phi}(x_{\max}) = 0.029799 \Rightarrow \text{Error } 2.2\%$

- $\phi^{**}(0) = \frac{1}{12}, \tilde{\phi}'(0) = \frac{1}{15}, \phi^{**}(1) = -\frac{1}{4}, \tilde{\phi}'(1) = -\frac{7}{30} \Rightarrow \text{Error } 20\%$

- Requirements on basic function $\zeta_i(x)$

- Linearly independent

- $\zeta_i(x) \in H^p(\Omega) \text{ or } \zeta_i(x) \in C^{p-1}(\Omega)$

- Characteristics of solution convergence

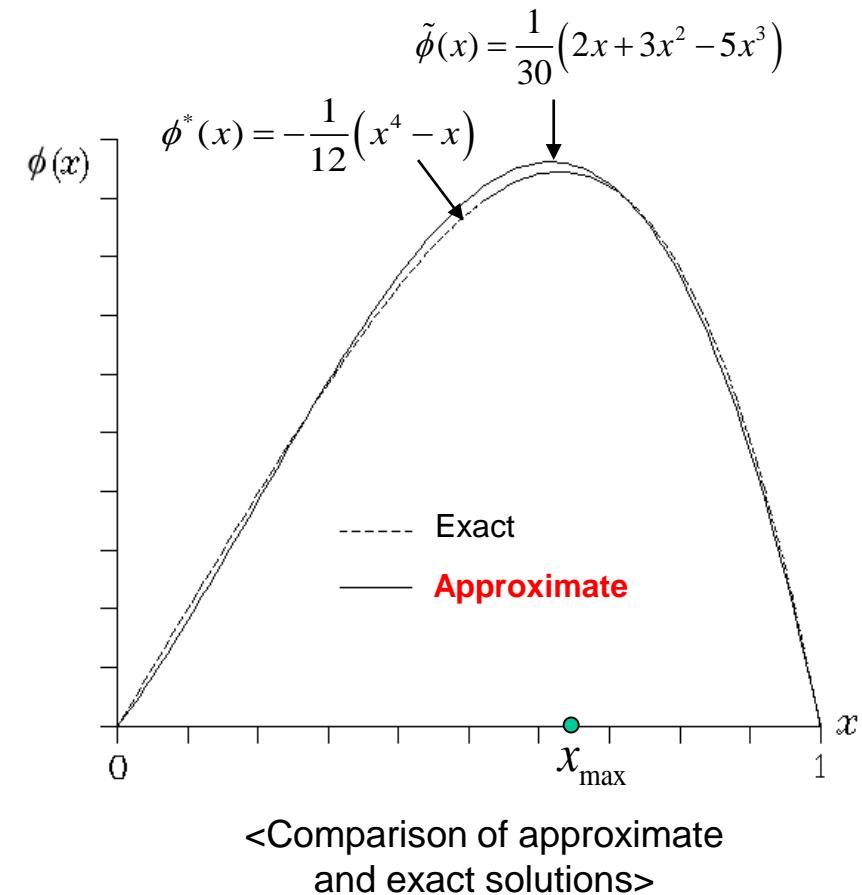
- Accuracy

$$\left(\sum_{i=1}^n C_i \zeta_i(x) \right) <$$

- Accuracy

$$\left(\sum_{i=1}^{n+1} C_i \zeta_i(x) \right)$$

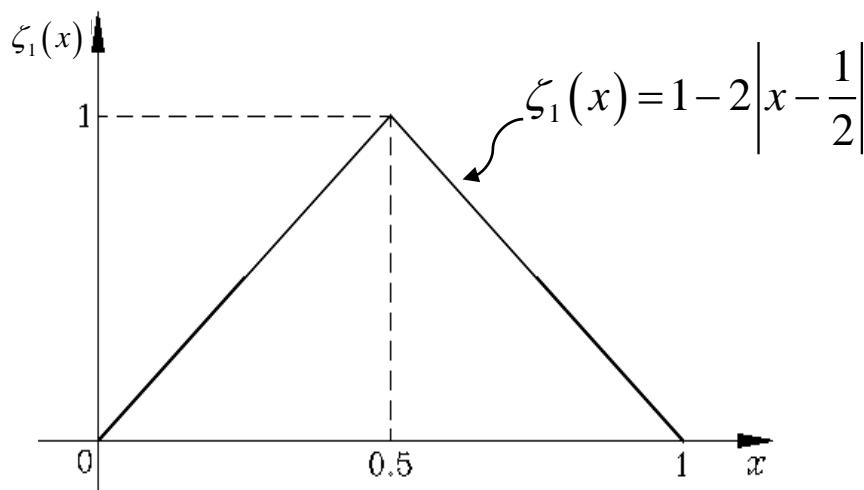
- $\tilde{\phi}(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n C_i \zeta_i(x) \Rightarrow \phi^*(x)$



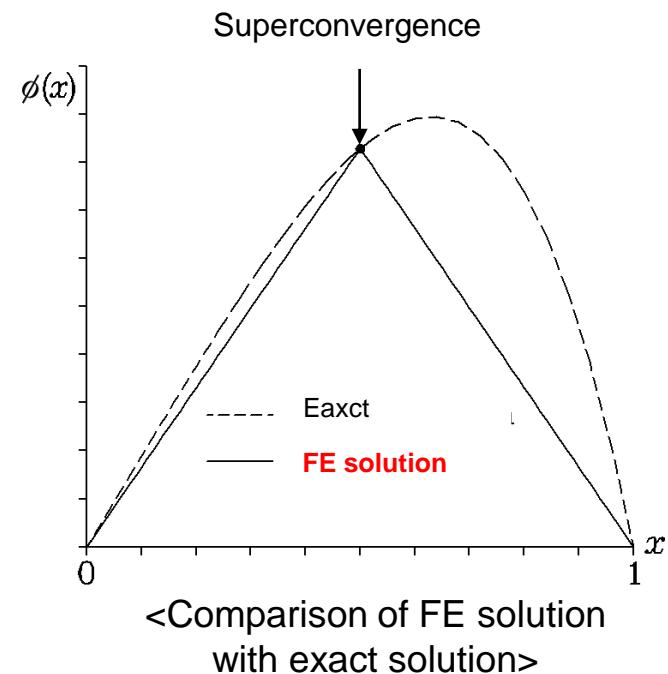


Basic idea of Finite Element Method (FEM)

- ⊙ Trial function : $\tilde{\phi}(x) = C_1 \left(1 - 2 \left| x - \frac{1}{2} \right| \right) = \begin{cases} 2C_1 x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2C_1 x(1-x) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$
- ⊙ Approximate solution : $\tilde{\phi}(x) = \frac{7}{192} - \frac{7}{96} \left| x - \frac{1}{2} \right|$



<Basic function = Interpolation function>

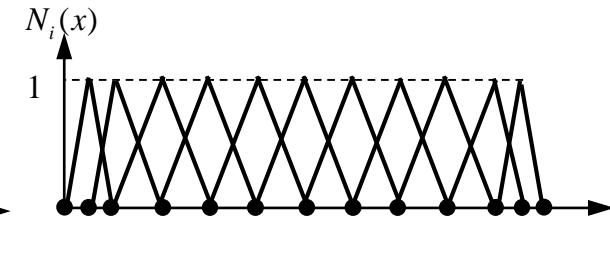
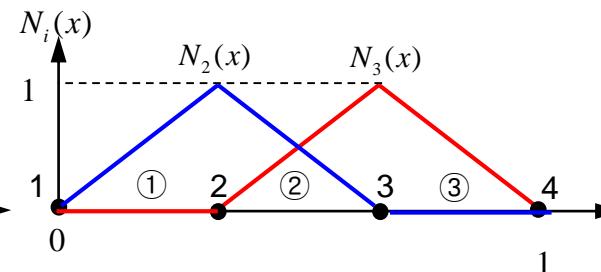
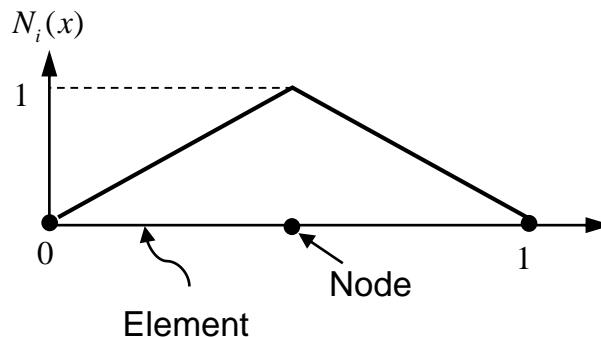


<Comparison of FE solution with exact solution>



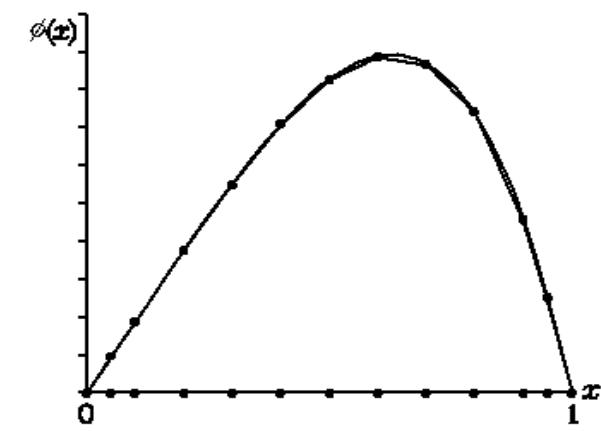
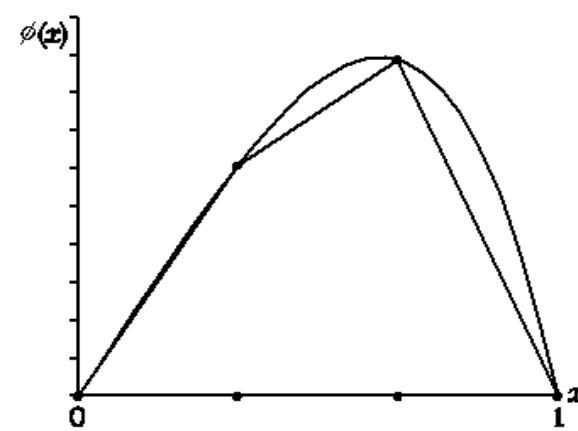
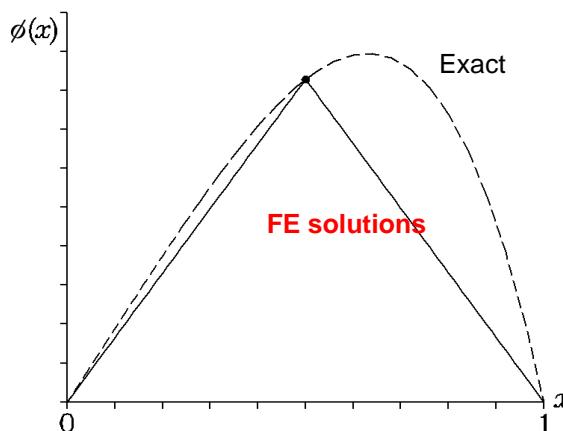
FE solutions with different FEA models

- FEM = Ritz or Galerkin method + FE discretization and interpolation (approximation)
- FE discretization and interpolation function: $N_i(x)$ Technique of making basic functions



- Finite element solutions

$$\begin{aligned} & \bullet \tilde{\phi}(x) = \phi_2 N_2(x) + \phi_3 N_3(x) \quad \text{in Finite Element Method} \\ & \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ & \bullet \tilde{\phi}(x) = C_1 \zeta_1(x) + C_2 \zeta_2(x) \quad \text{in Ritz or Galerkin method} \end{aligned}$$

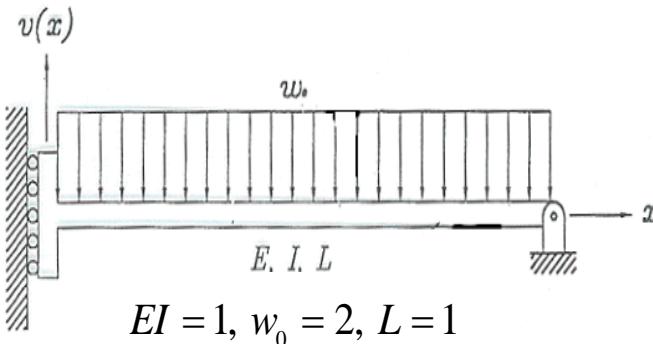




Ritz method to a beam deflection

- Definition of a beam deflection problem

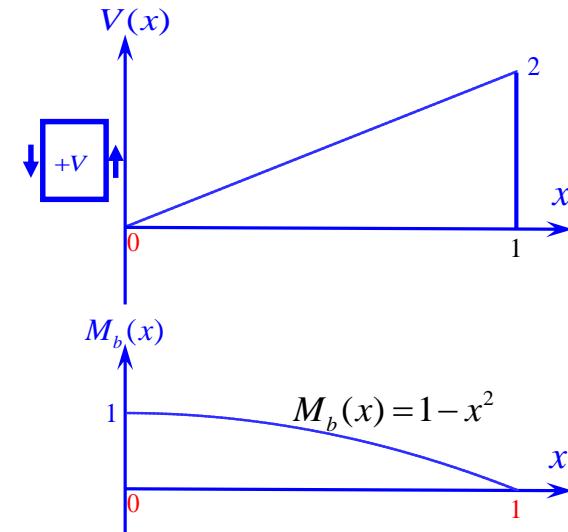
$$EIv''(x) = M_b(x), v'(0) = 0, v(L) = 0$$



- BVP based on beam theory:

$$\frac{d^2v}{dx^2} = 1 - x^2, \quad 0 < x < 1$$

$$v'(0) = 0, \quad v(1) = 0$$



- Variational principle:

$$\text{Extremize } F(v) = \int_0^1 \left[\left(\frac{dv}{dx} \right)^2 - 2(x^2 - 1)v(x) \right] dx$$

subject to $v(1) = 0$

- Ritz method

- It should satisfy essential boundary condition, $v(1) = 0$

- [Ex. 2.1] $\tilde{v}(x) = C_1(1-x)$

- [Ex. 2.2] $\tilde{v}(x) = C_1(1-x^2) + C_2(1-x)$

- [Ex. 2.3] $\tilde{v}(x) = C_1x^2(1-x) + C_2(1-x^2)$

- Solution:

$$v^*(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{5}{12}$$



Ritz method to a beam deflection

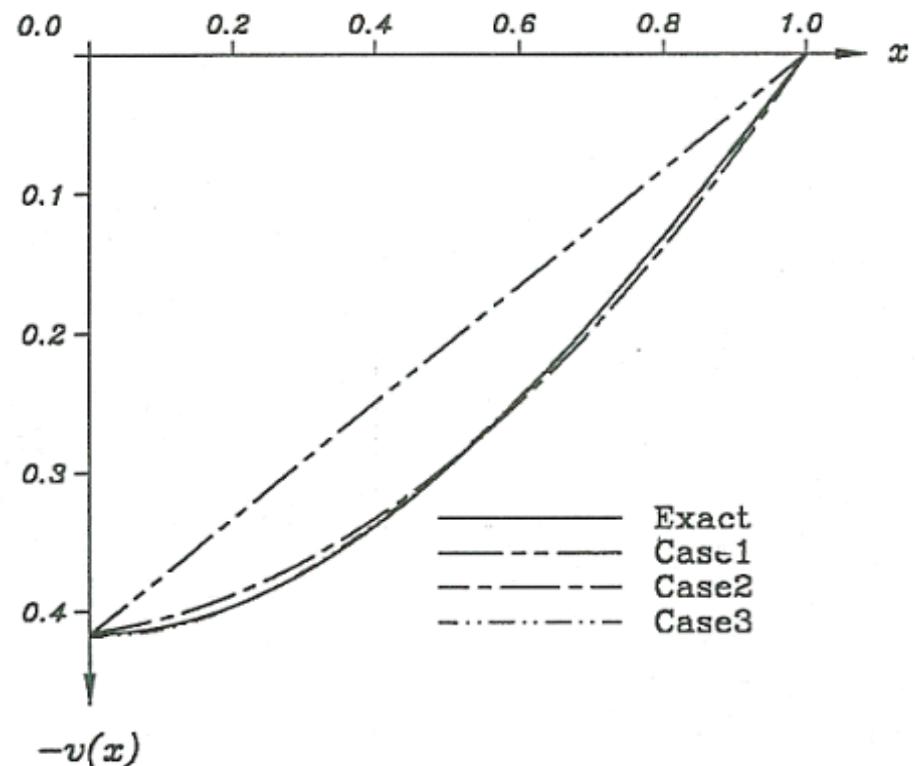
○ [Ex. 2.3]: $\tilde{v}(x) = C_1 x^2 (1-x) + C_2 (1-x^2)$

$$\begin{aligned}\tilde{F}(\tilde{v}) &= \int_0^1 \left[\{C_1(2x-3x^2) + C_2(-2x)\}^2 - 2(x^2-1)\{C_1(x^2-x^3) + C_2(1-x^2)\} \right] dx \\ &= \frac{5}{12}C_1^2 + \frac{4}{3}C_2^2 + \frac{1}{3}C_1C_2 + \frac{1}{10}C_1 + \frac{16}{15}C_2 = F(C_1, C_2)\end{aligned}$$

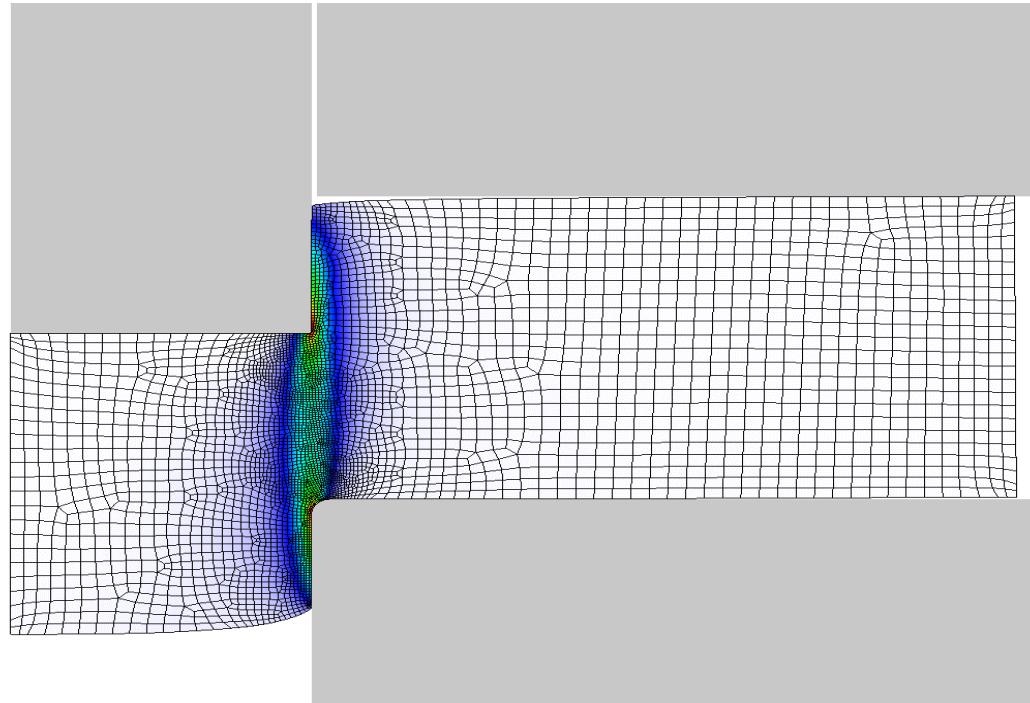
$$\begin{bmatrix} \frac{4}{15} & \frac{1}{3} \\ \frac{1}{3} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ -\frac{16}{15} \end{bmatrix}$$

$$\Rightarrow \tilde{v}(x) = -\frac{4}{27}x^3 - \frac{153}{270}x - \frac{113}{270}$$

Exact: $v^*(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{5}{12}$



5.2 FEM of Partial Differential Equations





Poisson's equation

► Poisson's equation

$$\frac{\partial}{\partial \mathbf{x}} \left(k \frac{\partial \phi}{\partial \mathbf{x}} \right) + \frac{\partial}{\partial \mathbf{y}} \left(k \frac{\partial \phi}{\partial \mathbf{y}} \right) + f(\mathbf{x}, \mathbf{y}) = 0$$

$$T = \bar{T} \text{ on } \mathbf{S}_T$$

2D = 3D

1D
$$-\frac{d^2 \phi}{dx^2} = x^2, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi(1) = 0$$

Extremize $F(\phi) = \frac{1}{2} \int_0^1 \left[\left(\frac{d\phi}{dx} \right)^2 - 2x^2 \phi(x) \right] dx$

subject to $\phi(0) = 0, \quad \phi(1) = 0$

$$\text{Extremize } F(\phi) = \frac{1}{2} \iint \left[k \left(\frac{\partial \phi}{\partial \mathbf{x}} \right)^2 + k \left(\frac{\partial \phi}{\partial \mathbf{y}} \right)^2 - 2f(\mathbf{x}, \mathbf{y})\phi(\mathbf{x}, \mathbf{y}) \right] d\mathbf{x} d\mathbf{y}$$

subject to $T = \bar{T}$ on \mathbf{S}_T

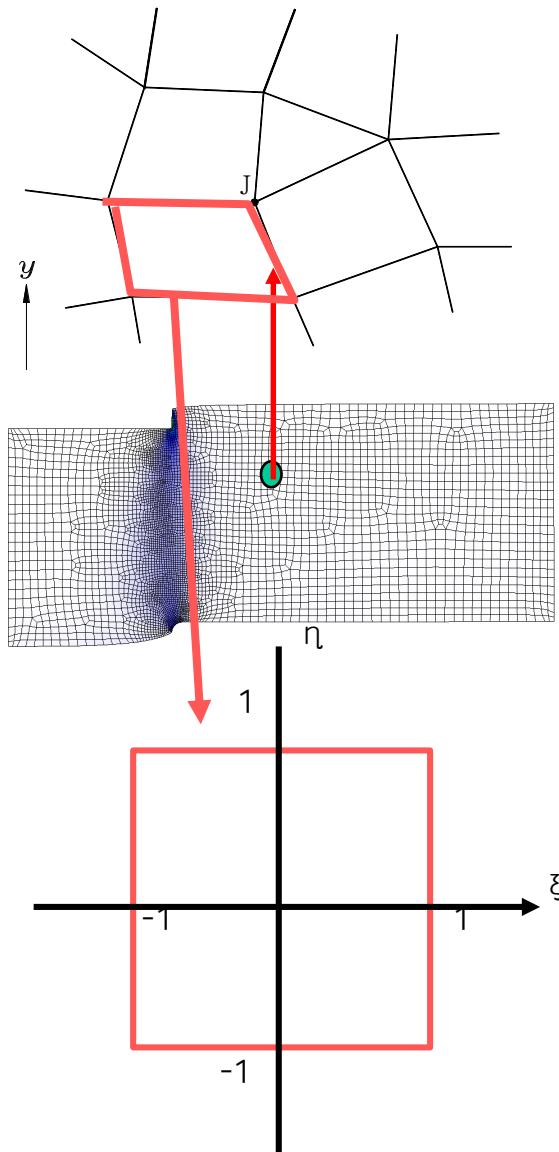
$$\tilde{\phi}(\mathbf{x}, \mathbf{y}) = \phi_1 \mathbf{N}_1(\mathbf{x}, \mathbf{y}) + \phi_2 \mathbf{N}_2(\mathbf{x}, \mathbf{y}) + \dots = \sum_J \phi_J \mathbf{N}_J \quad \mathbf{N}_J = ?, \quad \phi_J = ?$$

Nodal value

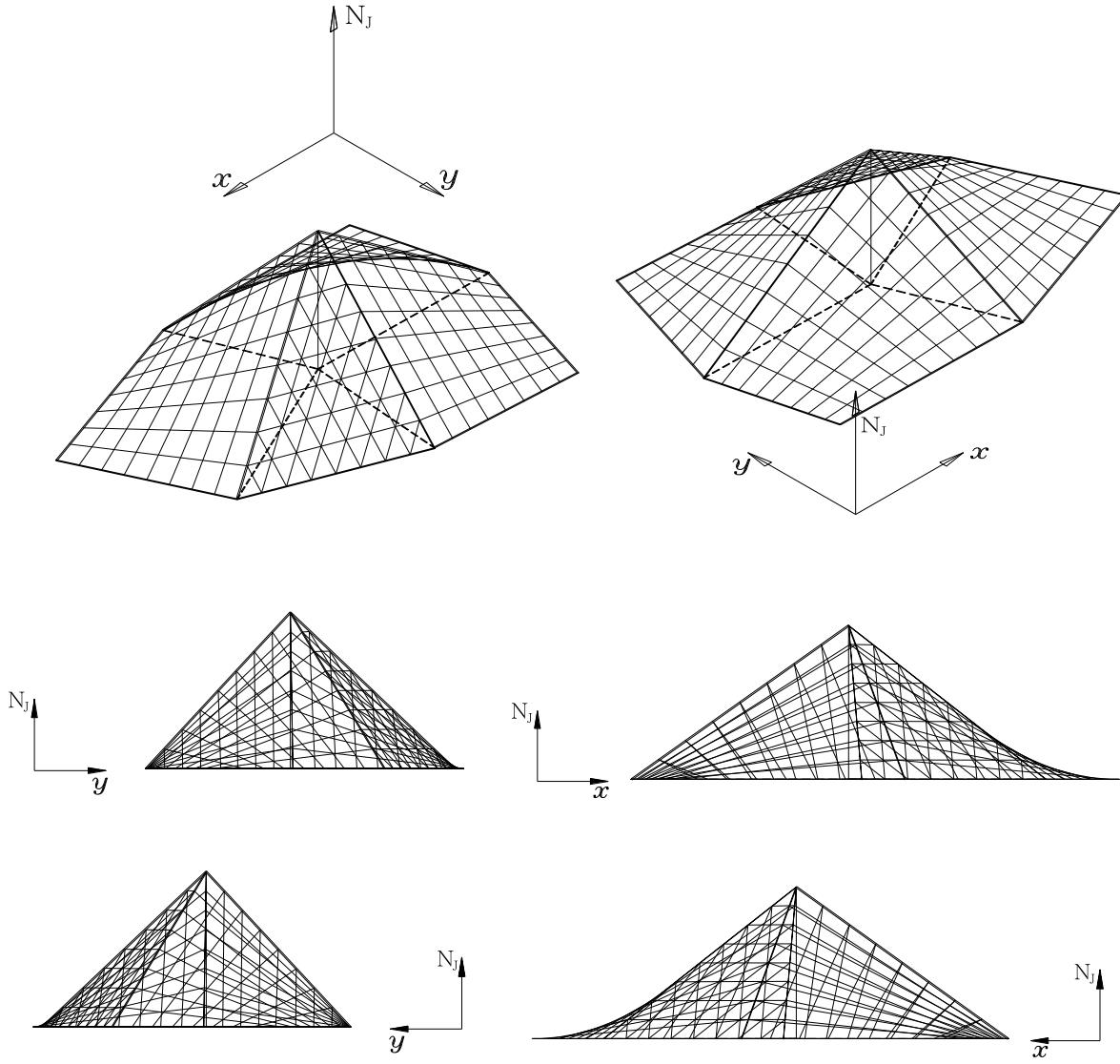


Mesh

Mesh system

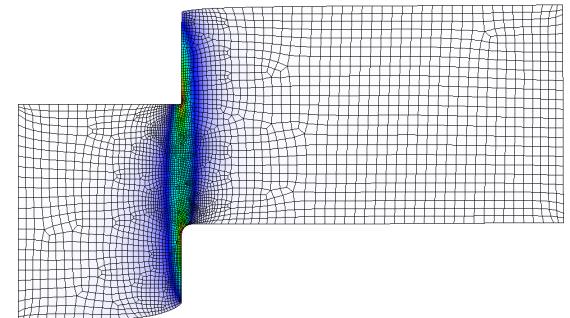
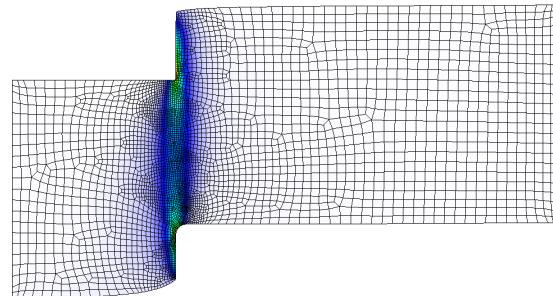
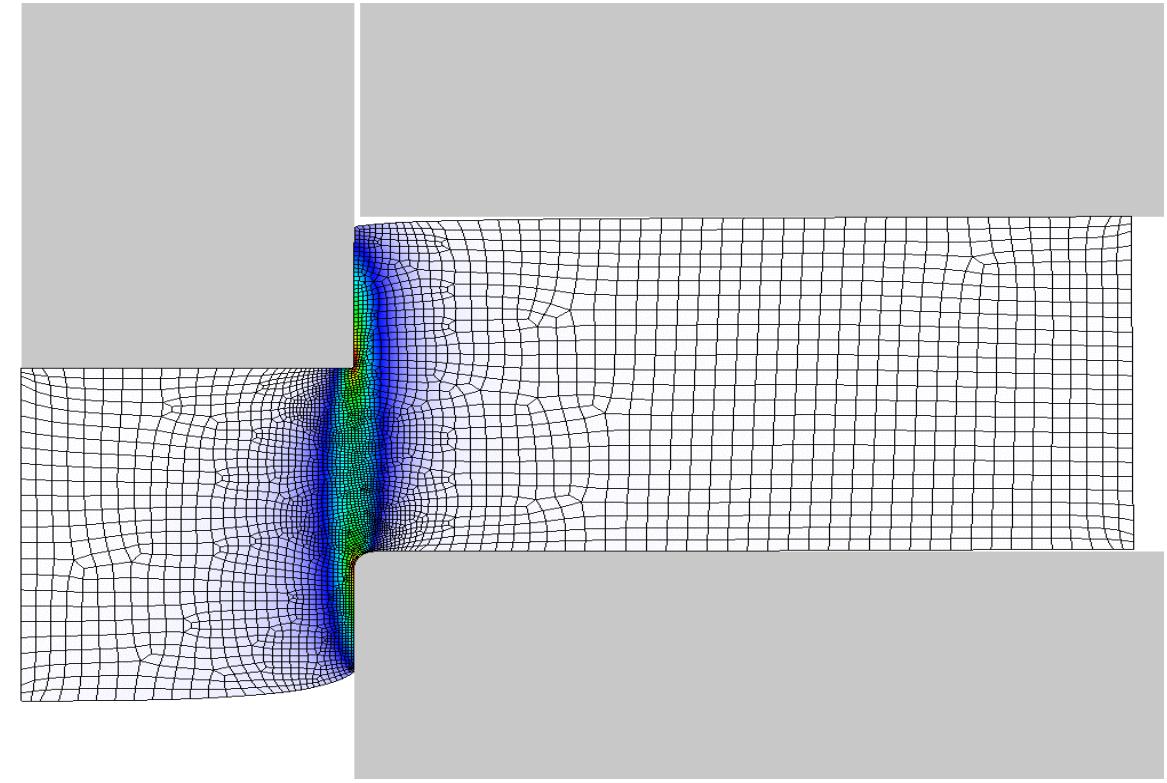
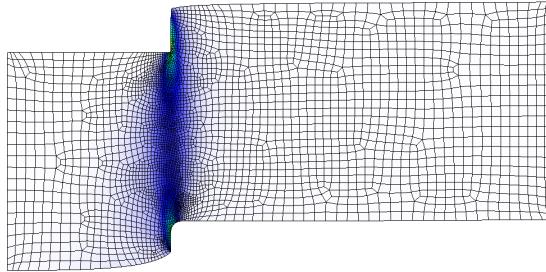
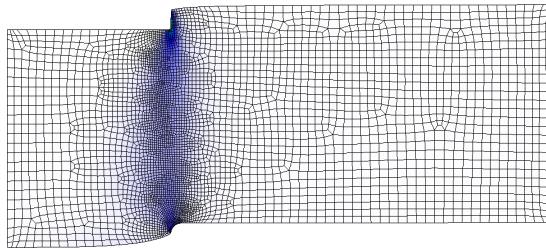
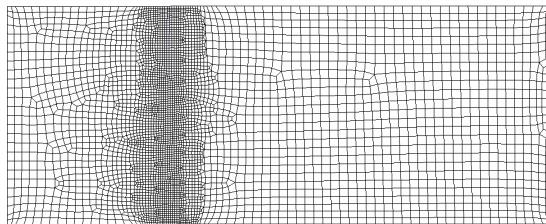


Interpolation function $N_J(x, y)$



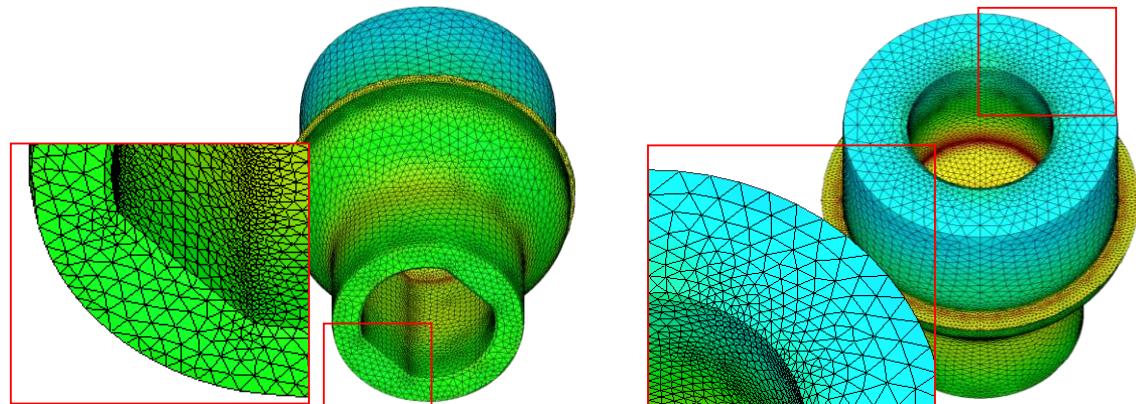
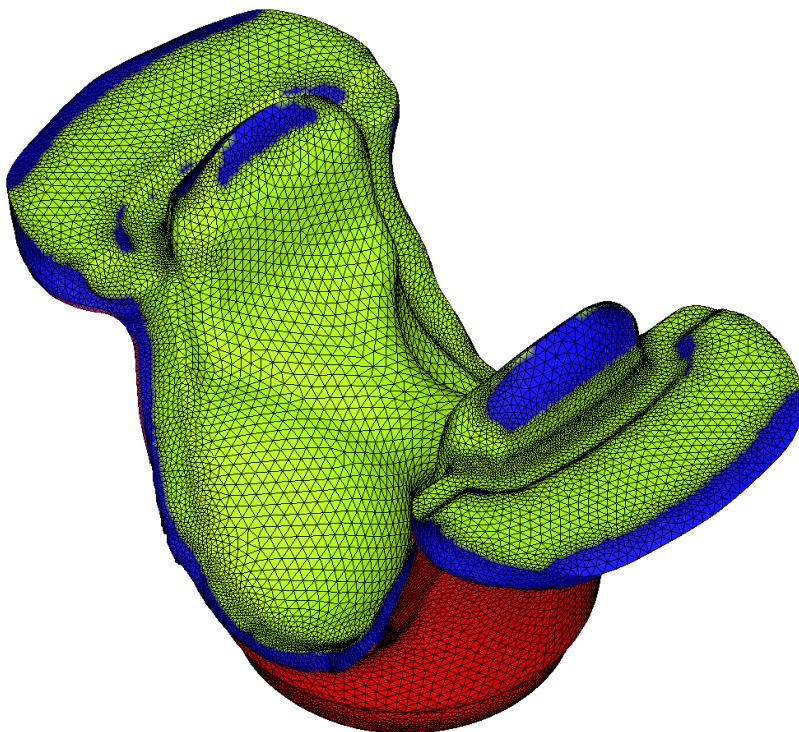


Intelligent remeshing of quadrilaterals





Intelligent remeshing of tetrahedrals

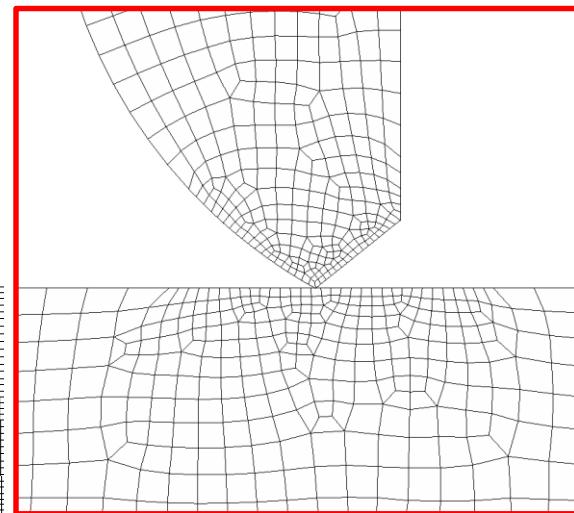
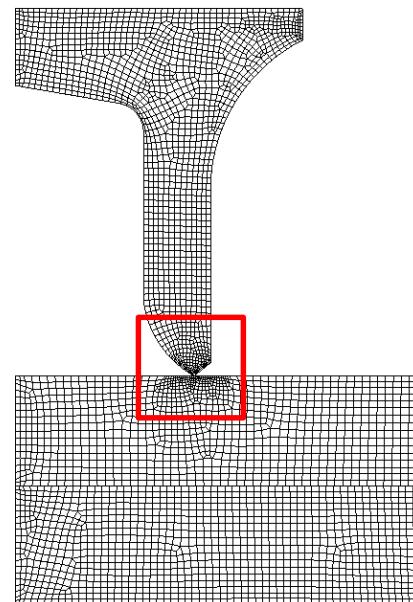
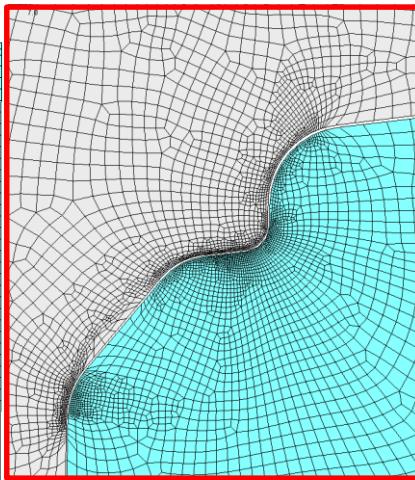
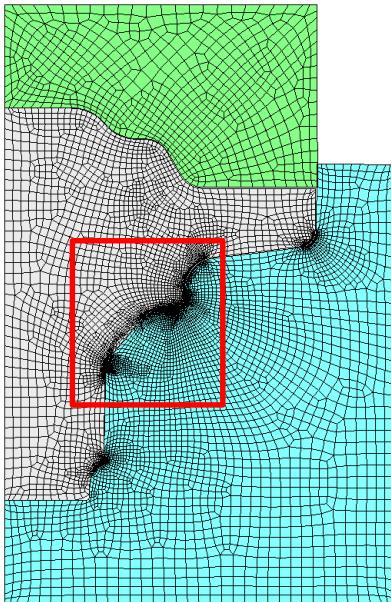


Accurate description
with minimum elements!

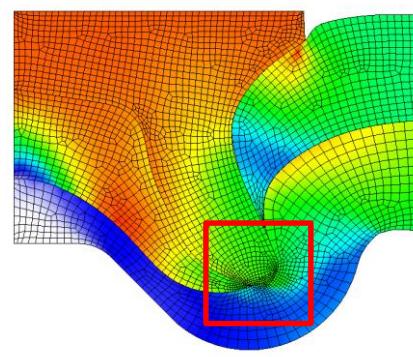
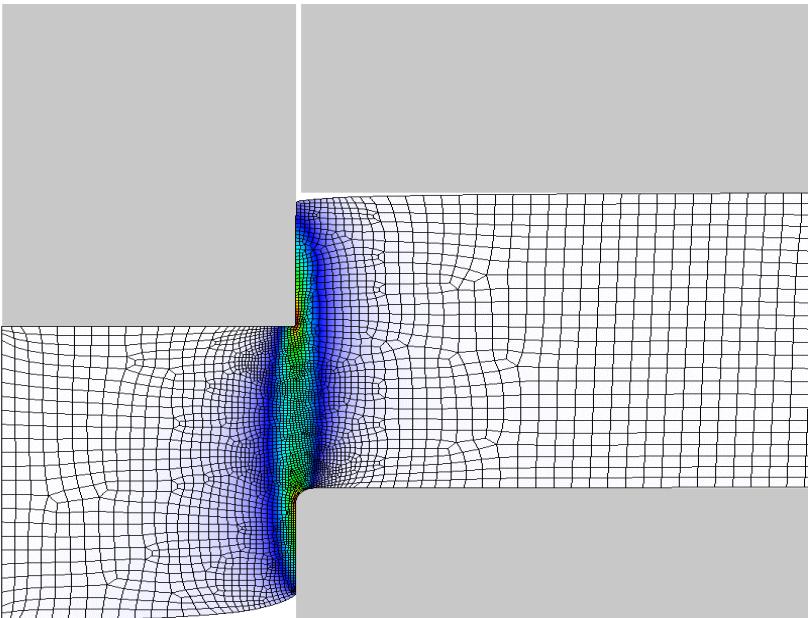
As number of elements increases,
that of remeshings does so much,
which deteriorates solution accuracy.



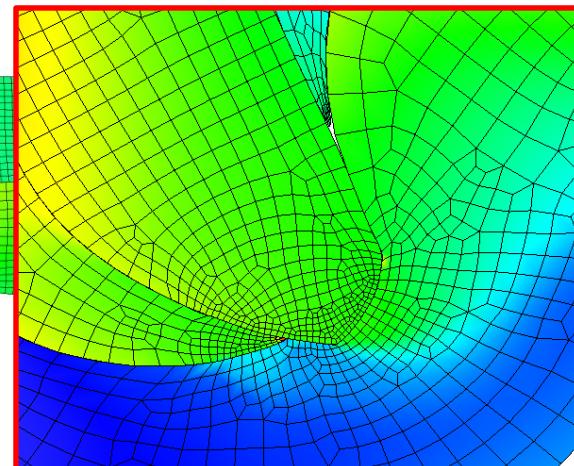
2D mesh systems with higher quality



Initial

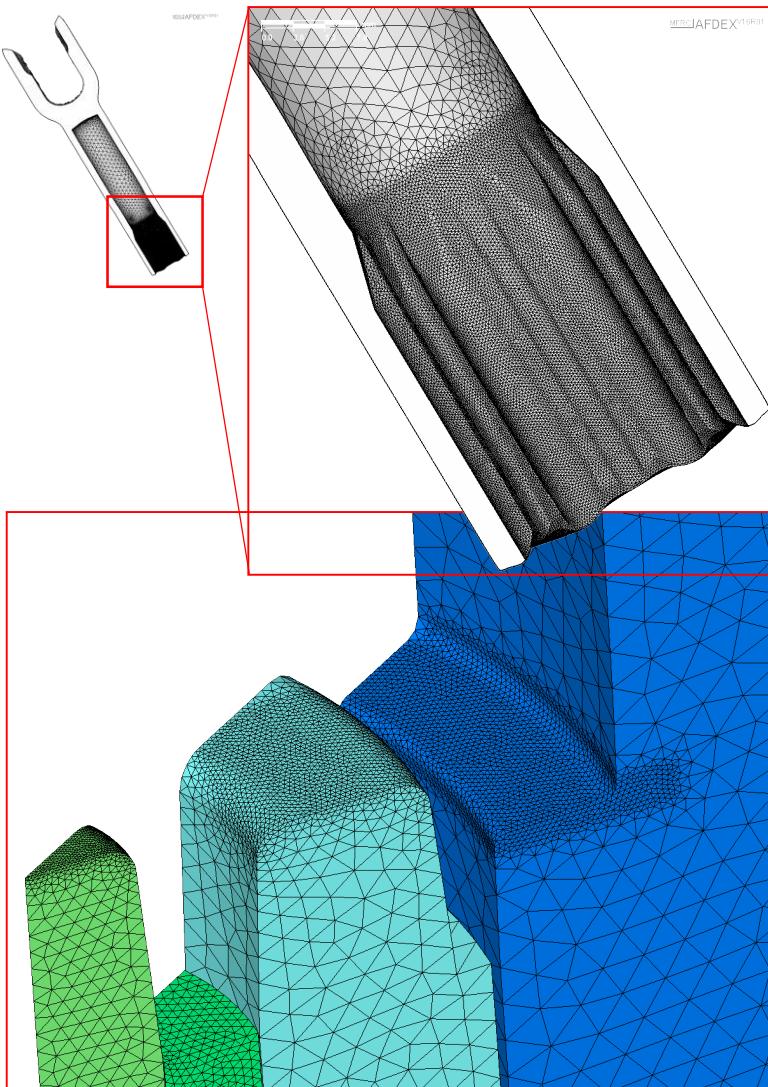


Final

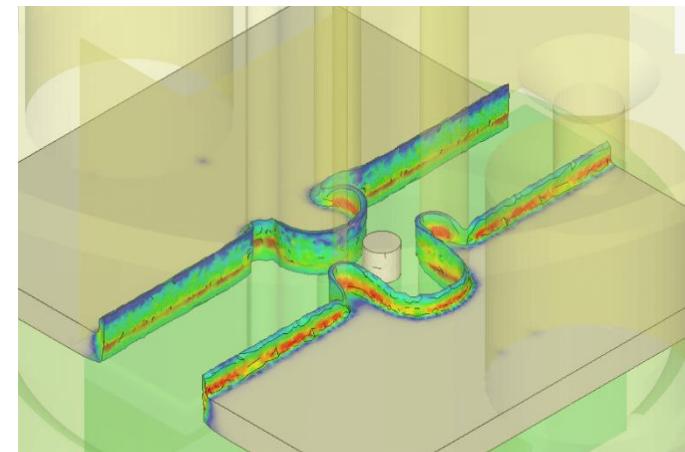




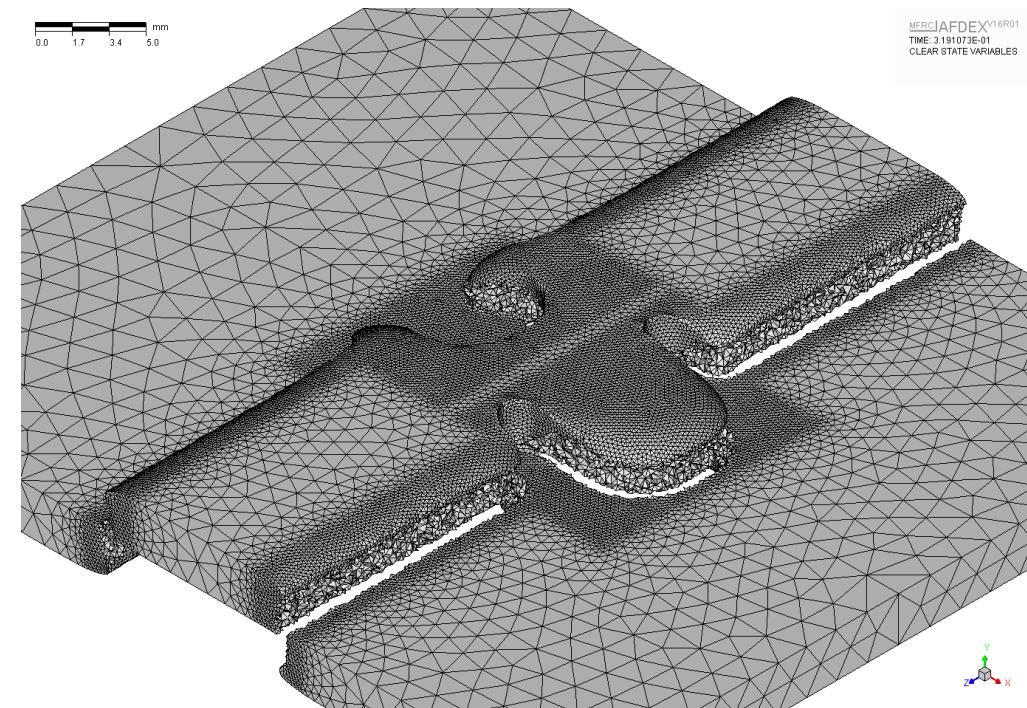
3D mesh systems with higher quality



0.0 1.7 3.4 5.0 mm

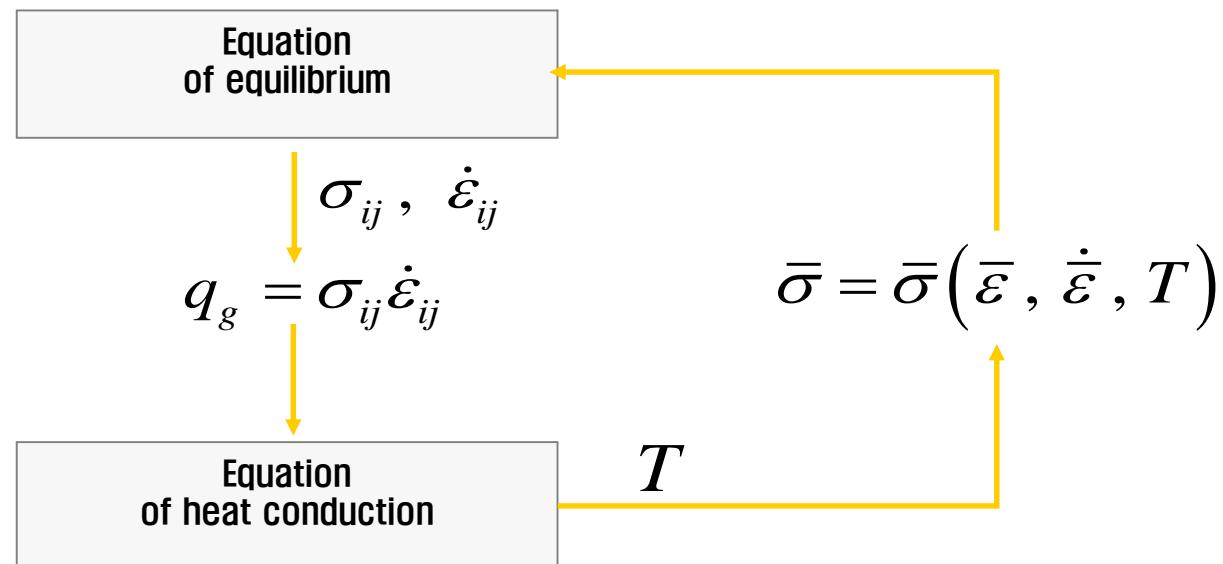


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CLEAR STATE VARIABLES





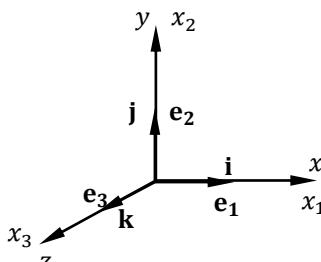
4.5 Finite element formulation





Tensor quantities and their indicial notation

Coordinate axis



$x, y, z - \text{axis} \Rightarrow x_1, x_2, x_3 - \text{axis}$

Mechanical quantities

$$(u_x, u_y) \Rightarrow (u_1, u_2)$$

$$\sigma_{xy} (\text{or } \tau_{xy}) \Rightarrow \sigma_{12}$$

Unit vector

$$\mathbf{i}, \mathbf{j}, \mathbf{k} \Rightarrow \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$$

Kronecker delta δ_{ij}

$$\delta_{ij} \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

First and second order

$$u_i = \mathbf{u} (\text{or } \vec{u})$$

$$\sigma_{ij} = [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Partial differentiation

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i}, \phi_{,ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}, v_{i,j} = \frac{\partial v_i}{\partial x_j}, \sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Summation

$$\sum_{j=1}^3 \sigma_{ij,j} + f_i = 0 \Rightarrow \sigma_{ij,j} + f_i = 0$$

Free index: once in a term (cannot be changed)

Dummy index: twice in a term

Divergence theorem

$$\int_V Q_{jk,m..m,i} dV = \int_S Q_{jk,m..m} n_i dS$$

n_i : Outwardly directed unit normal vector

Permutation symbol ε_{ijk}

$$\varepsilon_{ijk} \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ 1 & \text{if } (i, j, k) = (1, 2, 3) \text{ or } (2, 3, 1) = (3, 1, 2) \\ -1 & \text{if } (i, j, k) = (1, 3, 2) \text{ or } (2, 1, 3) = (3, 2, 1) \end{cases}$$

Examples

$$W = \mathbf{a} \bullet \mathbf{b} \Rightarrow W = a_i b_i$$

$$c = \mathbf{a} \times \mathbf{b} \Rightarrow c_i = \varepsilon_{ijk} a_j b_k$$

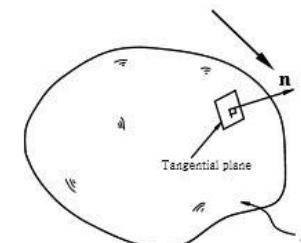
$$\text{grad } \phi \Rightarrow \phi_{,i}$$

$$\text{div } \mathbf{v} \Rightarrow v_{i,i}$$

$$\text{curl } \mathbf{v} \Rightarrow \varepsilon_{ijk} v_{k,j}$$

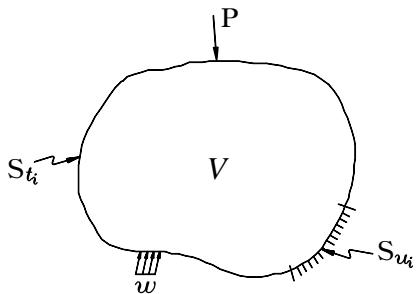
$$\nabla^2 \phi \Rightarrow \phi_{,ii}$$

Outwardly directed unit normal vector





Finite element analysis of 3D elastic problems of isotropic materials



- Equation of equilibrium
 $\sigma_{ij,j} + f_i = 0 \text{ in } V$
- displacement-strain relation
 $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
- stress-strain relation
 $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} - (3\lambda + 2\mu)\alpha\Delta T\delta_{ij}$
- Boundary condition
 $u_i = \bar{u}_i \text{ on } S_{u_i}$
 $t_i^{(n)} = \sigma_{ij}n_j = \bar{t}_i \text{ on } S_{t_i}$

Weak form

$$\int_V \sigma_{ij} \omega_{ij} dV - \int_{S_{t_i}} \bar{t}_i \omega_i dS - \int_V f_i \omega_i dV = 0$$

Weighted residual method
 $u_i' = \bar{u}_i'$, $\omega_i' = 0$ on $S_{u_i'}$

$$\omega_{ij} = \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

$$\begin{aligned} \sigma_{ij}\omega_{ij} &= \sigma_{xx}\omega_{xx} + \sigma_{yy}\omega_{yy} + \sigma_{zz}\omega_{zz} + 2\sigma_{xy}\omega_{xy} + 2\sigma_{yz}\omega_{yz} + 2\sigma_{zx}\omega_{zx} \\ &\equiv \sigma_i \beta_i \end{aligned}$$

Finite element equations

$$\begin{aligned} [\sigma_i] &= [D_{ij}] [\epsilon_j] && \text{Galerkin approximation} \\ \tilde{u}_i &= \sum_{I=1}^{3N} N_{il} U_I && \text{Shape function matrix} \\ [\sigma_i] &= [D_{ij}] [B_{jl}] [U_J] && \text{Stiffness matrix} \\ \sigma_{ij}\omega_{ij} &= \sigma_i \beta_i \\ &= (D_{ij} B_{jl} U_J) (B_{il} W_I) \\ &= W_I B_{jl} D_{ij} B_{jl} U_J \\ &= \mathbf{W}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{U} \end{aligned}$$

$$[\sigma] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}]^T$$

$$[\beta_i] = \begin{bmatrix} \omega_{xx} \\ \omega_{yy} \\ \omega_{zz} \\ 2\omega_{xy} \\ 2\omega_{yz} \\ 2\omega_{zx} \end{bmatrix} = \begin{bmatrix} \omega_{1,1} \\ \omega_{2,2} \\ \omega_{3,3} \\ \omega_{1,2} + \omega_{2,1} \\ \omega_{2,3} + \omega_{3,2} \\ \omega_{3,1} + \omega_{1,3} \end{bmatrix}$$

$$\beta_i = \begin{bmatrix} N_{1l,1} \\ N_{2l,2} \\ N_{3l,3} \\ N_{1l,2} + N_{2l,1} \\ N_{2l,3} + N_{3l,2} \\ N_{3l,1} + N_{1l,3} \end{bmatrix} W_I = B_{il} W_I$$

$$[B_{il}] = \begin{bmatrix} N_{1l,1} & 0 & 0 & N_{2l,1} & 0 & 0 & \cdots & N_{Nl,1} & 0 & 0 \\ 0 & N_{1l,2} & 0 & 0 & N_{2l,2} & 0 & \cdots & 0 & N_{Nl,2} & 0 \\ 0 & 0 & N_{1l,3} & 0 & 0 & N_{2l,3} & \cdots & 0 & 0 & N_{Nl,3} \\ N_{1l,2} & N_{1l,1} & 0 & N_{2l,2} & N_{2l,1} & 0 & \cdots & N_{Nl,2} & N_{Nl,1} & 0 \\ 0 & N_{1l,3} & N_{1l,2} & 0 & N_{2l,3} & N_{2l,2} & \cdots & 0 & N_{Nl,3} & N_{Nl,2} \\ N_{1l,3} & 0 & N_{1l,1} & N_{2l,3} & 0 & N_{2l,1} & \cdots & N_{Nl,3} & 0 & N_{Nl,1} \end{bmatrix}$$

Strain-displacement matrix

$$W_I \left[\int_V B_{il} D_{ij} B_{jl} dV U_J - \int_{S_{t_i}} \bar{t}_i N_{il} dS - \int_V f_i N_{il} dV \right] = 0$$

$$K_{IJ} = \int_V B_{il} D_{ij} B_{jl} dV$$

$$F_I = \int_{S_{t_i}} \bar{t}_i N_{il} dS + \int_V f_i N_{il} dV$$

Elastic matrix

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 1 & \gamma & 0 & 0 & 0 \\ \gamma & \gamma & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\gamma = \nu/(1-\nu), \quad \beta = (1-2\nu)/2(1-\nu)$$



Rigid-plastic finite element method

Weak form –

Penalty method

$$\int_V \sigma'_{ij} \omega_{ij} dV + \int_V K \dot{\varepsilon}_{ii} \omega_{jj} dV - \int_{S_{t_i}} \bar{t}_i \omega_i dS - \int_{S_c} \sigma_t \omega_t dS = 0$$

Penalty constant

Weak form – Lagrange multiplier method

Lagrange multiplier

Mixed formulation: knowns of velocity and pressure

$$\int_V \sigma'_{ij} \omega_{ij} dV - \int_V p \omega_{ii} dV - \int_V f_i \omega_i dV - \int_V v_{i,i} q dV - \int_{S_{t_i}} \bar{t}_i \omega_i dS - \int_{S_c} \sigma_t \omega_t dS = 0$$

$$\sigma'_{ij} \omega_{ij} = \sigma'_i \beta_i$$

$$= (D'_{ij} B_{jj} U_J) (B_{il} W_I)$$

$$= W_I B_{il} D'_{ij} B_{jj} U_J$$

$$= \mathbf{W}^T \mathbf{B}^T \mathbf{D}' \mathbf{B} \mathbf{U}$$

$$W_I \left[\int_V B_{il} D'_{ij} B_{jj} dV U_J - \int_V N_{il,i} H_M dV P_M - \int_{S_{t_i}} \bar{t}_i N_{il} dS - \int_V f_i N_{il} dV \right]$$

$$- Q_M \left[- \int_V N_{ij,i} H_M dV U_J \right] = 0$$

$$\sigma'_{ij} = \frac{2}{3} \frac{\bar{\sigma}}{\dot{\varepsilon}} \dot{\varepsilon}_{ij} = \frac{2}{3} \frac{\bar{\sigma}}{\sqrt{\frac{2}{3} \dot{\varepsilon}_{kl} \dot{\varepsilon}_{kl}}} \dot{\varepsilon}_{ij}$$

Non-linear

Numerical problem occurs in the elastic region.
Minimum allowable effective strain rate is needed.

Finite element equations

$$\begin{bmatrix} K'_{ij} & C_{IQ} \\ C_{MJ} & 0_{MQ} \end{bmatrix} \begin{bmatrix} U_J \\ P_Q \end{bmatrix} = \begin{bmatrix} F_I \\ G_M \end{bmatrix} \quad K'_{IJ} = \int_V B_{il} D'_{ij} B_{jj} dV$$

$$C_{IM} = - \int_V N_{il,i} H_M dV$$

- Over-constrained problem
- Reduced integration
- MINI-element



Finite element equations of heat conduction

Weak form

- Equation of heat conduction

$$\cdot (kT_{,i})_{,i} + q_g = \rho c \frac{\partial T}{\partial t} \text{ in } V$$

- Boundary condition

$$\cdot T = \bar{T} \text{ on } S_T$$

$$\cdot kT_{,i}n_i = -h_q(T - T_q) \text{ on } S_q$$

$$\cdot kT_{,i}n_i = -\sigma\varepsilon(T^4 - T_e^4) - h_e(T - T_e) \text{ on } S_e$$

$$\int_V (\rho c \frac{\partial T}{\partial t} \omega + kT_{,i} \omega_{,i} - Q\omega) dV - \int_{S_c} \left\{ q_f - h_c (T - T_c) \right\} \omega dS + \int_{S_q} h_q (T - T_w) \omega dS \\ + \int_{S_e} \left\{ \sigma\varepsilon (T^4 - T_e^4) + h_e (T - T_e) \right\} \omega dS = 0$$

$$Q = C_g \sigma_{ij} \dot{\varepsilon}_{ij}$$

Finite element equations

$$T = N_I T_I$$

$$\omega = N_I W_I$$

$$\frac{\partial T}{\partial t} = N_I \dot{T}_I$$

$$\dot{T}_I^{i+\theta} = \frac{T_I^{i+\theta} - T_I^i}{\theta \Delta t} = \frac{T_I^{i+1} - T_I^i}{\Delta t}$$

$$t^{i+\theta} = t^i + \theta \Delta t$$

$$T_I^{i+1} = T_I^i + \dot{T}_I^{i+\theta} \Delta t$$

$$W_I \left[\int_V \left(\rho c N_J \dot{T}_J N_I + k N_{J,i} N_{I,i} T_J - Q N_I \right) dV - \int_{S_c} \left\{ q_f - h_c (N_J T_J - T_c) \right\} N_I dS \right. \\ \left. + \int_{S_q} h_q (N_J T_J - T_w) N_I dS + \int_{S_e} \left\{ \sigma\varepsilon ((\sum N_J T_J)^4 - T_e^4) + h_e (N_J T_J - T_e) \right\} N_I dS \right] = 0$$

$$C_{IJ} \dot{T}_J + (K_{IJ}^0 + K_{IJ}^1 + K_{IJ}^2 + K_{IJ}^3 + K_{IJ}^4) T_J = Q_I^1 + Q_I^2 + Q_I^3 + Q_I^4 \\ (C_{IJ}^{i+\theta} / (\theta \Delta t) + K_{IJ}^{0,i+\theta} + K_{IJ}^{1,i+\theta} + K_{IJ}^{2,i+\theta} + K_{IJ}^{3,i+\theta} + K_{IJ}^{4,i+\theta}) T_J^{i+\theta} \\ = Q_I^{1,i+\theta} + Q_I^{2,i+\theta} + Q_I^{3,i+\theta} + Q_I^{4,i+\theta} + C_{IJ}^{i+\theta} T_J^i / (\theta \Delta t)$$

$$C_{IJ} = \int_V \rho c N_I N_J dV$$

$$K_{IJ}^0 = \int_V k N_{I,i} N_{J,i} dV$$

$$K_{IJ}^3 = \int_{S_e} h_e N_I N_J dS$$

$$Q_I^1 = \int_V Q N_I dV$$

$$K_{IJ}^1 = \int_{S_c} h_c N_I N_J dS$$

$$K_{IJ}^2 = \int_{S_q} h_q N_I N_J dS$$

$$Q_I^4 = \int_{S_e} (\sigma\varepsilon T_e^4 + h_e T_e) N_I dS$$

$$K_{IJ}^4 = \int_{S_e} \sigma\varepsilon (\sum N_I N_J)^3 N_I N_J dS$$

$$Q_I^2 = \int_{S_c} (q_f + h_c T_c) N_I dS$$

$$Q_I^3 = \int_{S_q} h_q T_w N_I dS$$

